An electromagnetic calculation of electric field mapping that finds very unexpected results

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Dynamical evolution toward electrostatic equilibrium (or not) happens through waves.

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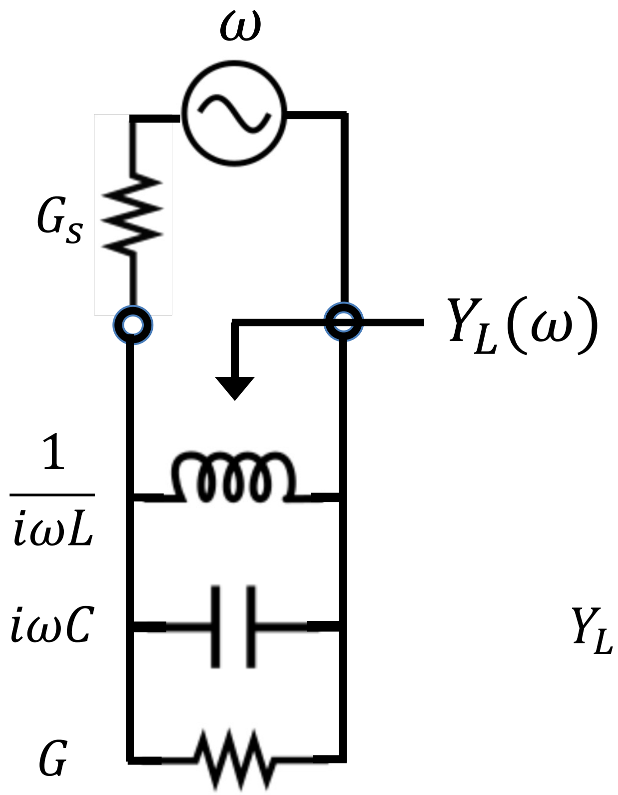
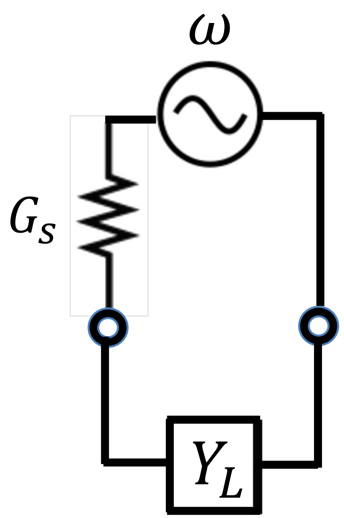
Electrical Engineering Basics, Transmission Lines

lumped element
AC circuit in
steady state

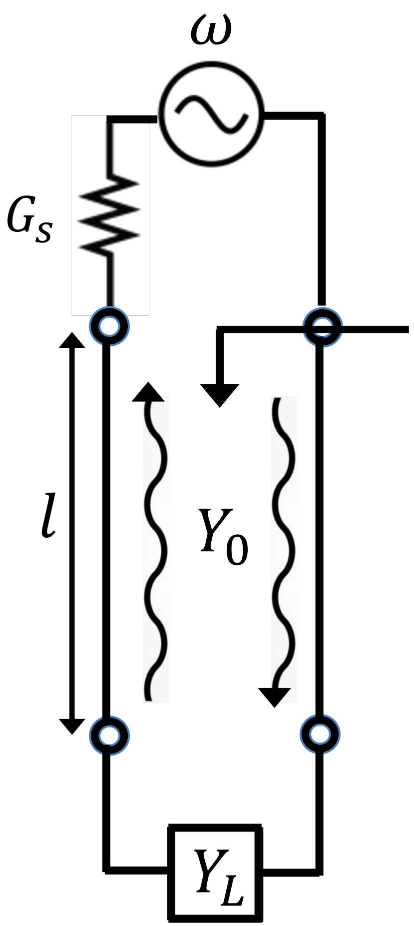


short hand



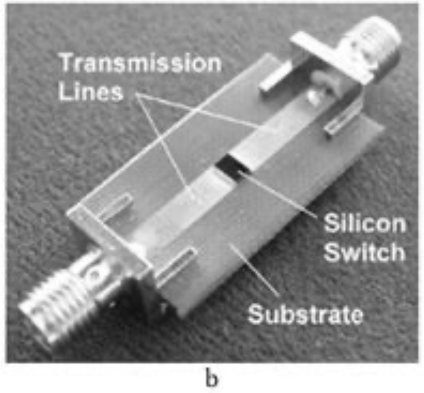
$$Y_L = G + i\omega C + \frac{1}{i\omega L}$$

space between
source and load

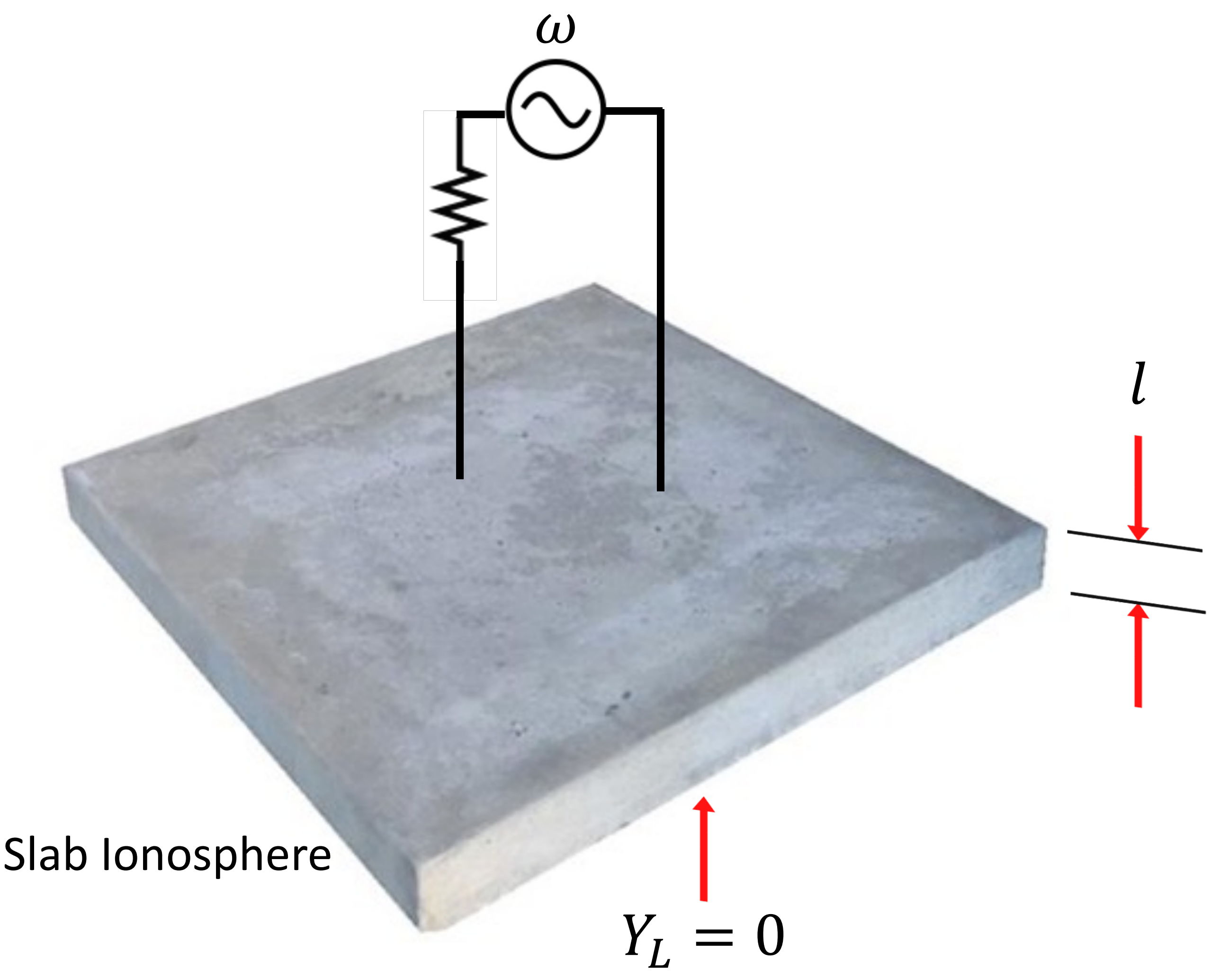


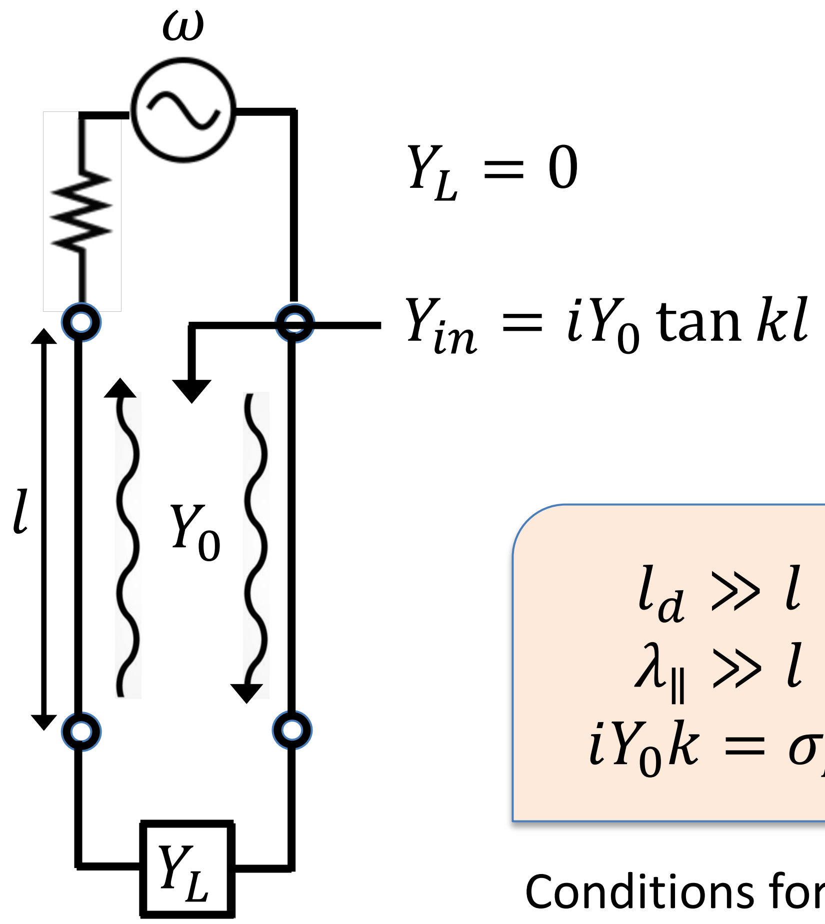
Transmission Line formula:
Superpose the waves, match Y_L

$$Y_{in} = Y_0 \frac{iY_0 \tan kl + Y_L}{iY_L \tan kl + Y_0}$$



Gedankenexperiment: Slab Ionosphere





$$l_d \gg l$$
$$\lambda_{\parallel} \gg l$$
$$iY_0 k = \sigma_P$$

Conditions for electrostatic theory

Electromagnetic 5-Moment Fluid Equations, Matrix Form:

$$\frac{\partial \vec{X}}{\partial t} - iH_5\vec{X} = \vec{F}(t)$$

Nonlinear terms we will drop, like electrostatic theory.

- momentum equations
- continuity equations
- energy equations
- Maxwell equations
- electrons and ions

$$H_5 = i\left[\begin{array}{cccccccccccccccc}
\nu_{en}+\nu_{ei}-i\vec{k}\cdot\vec{V}_{e0}+\gamma_x V_{e0x} & \Omega_{e0}\hat{B}_z+\gamma_y V_{e0x} & -\Omega_{e0}\hat{B}_y+\gamma_z V_{e0x} & -\nu_{ei}\sqrt{\frac{m_e}{m_i}} & 0 & 0 & \omega_e & 0 & 0 & 0 & \frac{-V_{e0z}}{c}\omega_e & \frac{V_{e0y}}{c}\omega_e & 0 & 0 & \frac{m_i}{m_e}\frac{k_B T_n|\vec{k}|}{m_e\omega_e}(-ik_x-\gamma_x) & 0 \\
-\Omega_{e0}\hat{B}_z+\gamma_x V_{e0y} & \nu_{en}+\nu_{ei}-i\vec{k}\cdot\vec{V}_{e0}+\gamma_y V_{e0y} & \Omega_{e0}\hat{B}_x+\gamma_z V_{e0y} & 0 & -\nu_{ei}\sqrt{\frac{m_e}{m_i}} & 0 & 0 & \omega_e & 0 & \frac{V_{e0z}}{c}\omega_e & 0 & \frac{-V_{e0x}}{c}\omega_e & 0 & 0 & \frac{m_i}{m_e}\frac{k_B T_n|\vec{k}|}{m_e\omega_e}(-ik_y-\gamma_y) & 0 \\
\Omega_{e0}\hat{B}_y+\gamma_x V_{e0z} & -\Omega_{e0}\hat{B}_x+\gamma_y V_{e0z} & \nu_{en}+\nu_{ei}-i\vec{k}\cdot\vec{V}_{e0}+\gamma_z V_{e0z} & 0 & 0 & -\nu_{ei}\sqrt{\frac{m_e}{m_i}} & 0 & 0 & \omega_e & \frac{-V_{e0y}}{c}\omega_e & \frac{V_{e0x}}{c}\omega_e & 0 & 0 & 0 & \frac{m_i}{m_e}\frac{k_B T_n|\vec{k}|}{m_e\omega_e}(-ik_z-\gamma_z) & 0 \\
-\nu_{ie}\sqrt{\frac{m_i}{m_e}} & 0 & 0 & \nu_{in}+\nu_{ie}-i\vec{k}\cdot\vec{V}_{i0}+\gamma_x V_{i0x} & -\Omega_{i0}\hat{B}_z+\gamma_y V_{i0x} & \Omega_{i0}\hat{B}_y+\gamma_z V_{i0x} & -\omega_i & 0 & 0 & 0 & \frac{V_{i0z}}{c}\omega_i & \frac{-V_{i0y}}{c}\omega_i & 0 & 0 & 0 & \frac{k_B T_n|\vec{k}|}{m_e\omega_i}(-ik_x-\gamma_x) \\
0 & -\nu_{ie}\sqrt{\frac{m_i}{m_e}} & 0 & \Omega_{i0}\hat{B}_z+\gamma_x V_{i0y} & \nu_{in}+\nu_{ie}-i\vec{k}\cdot\vec{V}_{i0}+\gamma_y V_{i0y} & -\Omega_{i0}\hat{B}_x+\gamma_z V_{i0y} & 0 & -\omega_i & 0 & \frac{-V_{i0z}}{c}\omega_i & 0 & \frac{V_{i0x}}{c}\omega_i & 0 & 0 & 0 & \frac{k_B T_n|\vec{k}|}{m_e\omega_i}(-ik_y-\gamma_y) \\
0 & 0 & -\nu_{ie}\sqrt{\frac{m_i}{m_e}} & -\Omega_{i0}\hat{B}_y+\gamma_x V_{i0z} & \Omega_{i0}\hat{B}_x+\gamma_y V_{i0z} & \nu_{in}+\nu_{ie}-i\vec{k}\cdot\vec{V}_{i0}+\gamma_z V_{i0z} & 0 & 0 & -\omega_i & \frac{V_{i0y}}{c}\omega_i & \frac{-V_{i0x}}{c}\omega_i & 0 & 0 & 0 & 0 & \frac{k_B T_n|\vec{k}|}{m_e\omega_i}(-ik_z-\gamma_z) \\
-\omega_e & 0 & 0 & \omega_i & 0 & 0 & 0 & 0 & 0 & 0 & -cik_z & cik_y & -\frac{m_i}{m_e}|\vec{k}|V_{e0x} & \frac{m_i}{m_e}|\vec{k}|V_{i0x} & 0 & 0 \\
0 & -\omega_e & 0 & 0 & \omega_i & 0 & 0 & 0 & 0 & cik_z & 0 & -cik_x & -\frac{m_i}{m_e}|\vec{k}|V_{e0y} & \frac{m_i}{m_e}|\vec{k}|V_{i0y} & 0 & 0 \\
0 & 0 & -\omega_e & 0 & 0 & \omega_i & 0 & 0 & 0 & -cik_y & cik_x & 0 & -\frac{m_i}{m_e}|\vec{k}|V_{e0z} & \frac{m_i}{m_e}|\vec{k}|V_{i0z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & cik_z & -cik_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -cik_z & 0 & cik_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & cik_y & -cik_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{m_e}{m_i}\frac{\omega_e}{|\vec{k}|}(\gamma_x-ik_x) & \frac{m_e}{m_i}\frac{\omega_e}{|\vec{k}|}(\gamma_y-ik_y) & \frac{m_e}{m_i}\frac{\omega_e}{|\vec{k}|}(\gamma_z-ik_z) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vec{\gamma}\cdot\vec{V}_{e0}-i\vec{k}\cdot\vec{V}_{e0}+\xi\bar{n}_0 & \xi\bar{n}_0 & 0 & 0 \\
0 & 0 & 0 & \frac{m_e}{m_i}\frac{\omega_i}{|\vec{k}|}(\gamma_x-ik_x) & \frac{m_e}{m_i}\frac{\omega_i}{|\vec{k}|}(\gamma_y-ik_y) & \frac{m_e}{m_i}\frac{\omega_i}{|\vec{k}|}(\gamma_z-ik_z) & 0 & 0 & 0 & 0 & 0 & 0 & \xi\bar{n}_0 & \vec{\gamma}\cdot\vec{V}_{i0}-i\vec{k}\cdot\vec{V}_{i0}+\xi\bar{n}_0 & 0 & 0 \\
\frac{m_e}{m_i}A_{eix} & \frac{m_e}{m_i}A_{eiy} & \frac{m_e}{m_i}A_{eiz} & \frac{m_e}{m_i}C_{eix} & \frac{m_e}{m_i}C_{eiy} & \frac{m_e}{m_i}C_{eiz} & 0 & 0 & 0 & 0 & 0 & 0 & D_{ei} & \frac{2m_e\nu_{ei}P_{io}}{(m_e+m_i)k_B T_n\bar{n}_0} & L_{ei} & -2\frac{m_e\nu_{ei}}{m_e+m_i} \\
\frac{m_e}{m_i}C_{iex} & \frac{m_e}{m_i}C_{iey} & \frac{m_e}{m_i}C_{iez} & \frac{m_e}{m_i}A_{iex} & \frac{m_e}{m_i}A_{iey} & \frac{m_e}{m_i}A_{iez} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2m_i\nu_{ie}P_{eo}}{(m_e+m_i)k_B T_n\bar{n}_0} & D_{ie} & -2\frac{m_i\nu_{ie}}{m_e+m_i} & L_{ie}
\end{array}\right], \text{ where}$$

$$A_{\alpha\beta} = \frac{\vec{\gamma}}{|\vec{k}|}\frac{P_{\alpha 0}\omega_\alpha}{k_B T_n\bar{n}_0} - \frac{5}{3}\frac{i\vec{k}}{|\vec{k}|}\frac{P_{\alpha 0}\omega_\alpha}{k_B T_n\bar{n}_0} - \frac{4}{3}\frac{(\vec{V}_{\alpha 0}-\vec{V}_n)\frac{\omega_\alpha}{|\vec{k}|}}{k_B T_n/m_\alpha}\frac{m_n\nu_{\alpha n}}{m_\alpha+m_n} - \frac{4}{3}\frac{(\vec{V}_{\alpha 0}-\vec{V}_{\beta 0})\frac{\omega_\alpha}{|\vec{k}|}}{k_B T_n/m_\alpha}\frac{m_\beta\nu_{\alpha\beta}}{m_\alpha+m_\beta},$$

$$C_{\alpha\beta} = \frac{4}{3}\frac{(\vec{V}_{\alpha 0}-\vec{V}_{\beta 0})\frac{\omega_\beta}{|\vec{k}|}}{k_B T_n/m_\alpha}\frac{m_\beta\nu_{\alpha\beta}}{m_\alpha+m_\beta},$$

$$D_{\alpha\beta} = -2\frac{m_\alpha\nu_{\alpha n}}{m_\alpha+m_n} - 2\frac{m_\alpha\nu_{\alpha\beta}P_{\beta 0}}{(m_\alpha+m_\beta)k_B T_n\bar{n}_0} - \frac{2}{3}\frac{m_n\nu_{\alpha n}}{m_\alpha+m_n}\frac{\left|\vec{V}_{\alpha 0}-\vec{V}_n\right|^2}{k_B T_n/m_\alpha} - \frac{2}{3}\frac{m_\beta\nu_{\alpha\beta}}{m_\alpha+m_\beta}\frac{\left|\vec{V}_{\alpha 0}-\vec{V}_{\beta 0}\right|^2}{k_B T_n/m_\alpha},$$

$$L_{\alpha\beta} = -i\vec{k}\cdot\vec{V}_{\alpha 0} + \frac{5}{3}\left(\vec{\gamma}\cdot\vec{V}_{\alpha 0}\right) + 2\frac{m_\alpha\nu_{\alpha n}}{m_\alpha+m_n} + 2\frac{m_\alpha\nu_{\alpha\beta}}{m_\alpha+m_\beta}.$$

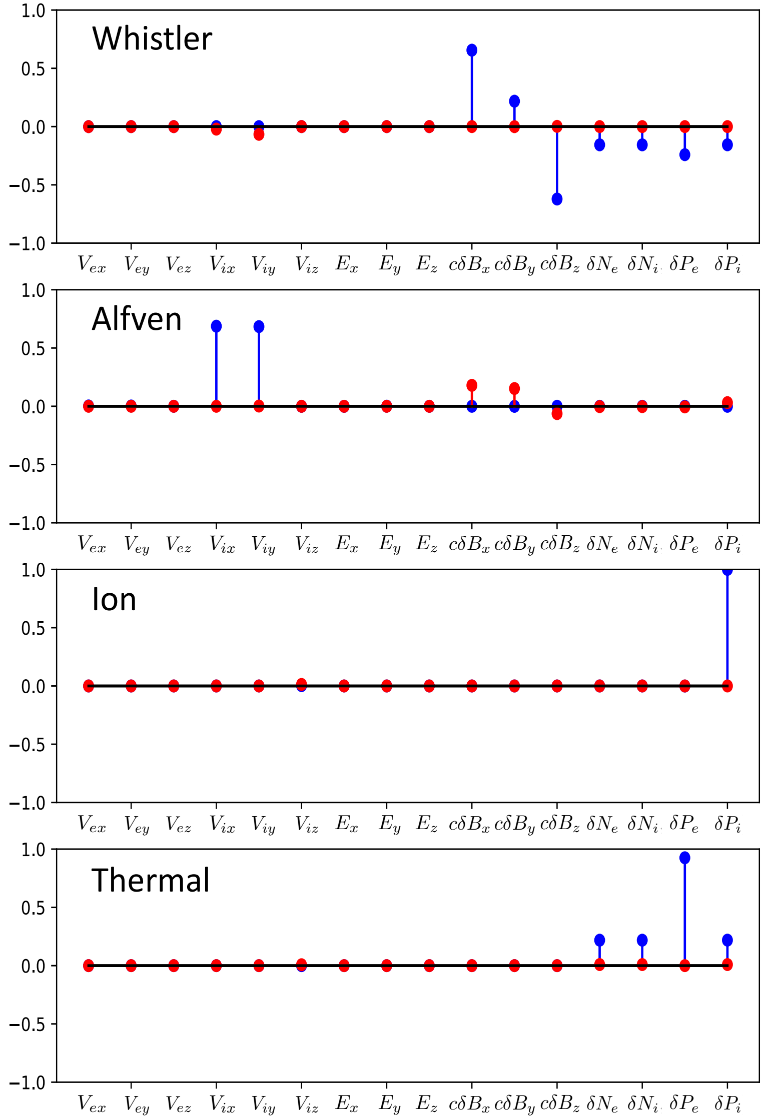
arbitrary coefficients — eigenvectors — eigenvalues

$$\vec{X}\left(t,\vec{k}\right) = \sum_{j=1}^{16} a_{0j}(\vec{k})\,\vec{h}_j(\vec{k})\,\mathrm{e}^{i\omega_j(\vec{k})t}$$

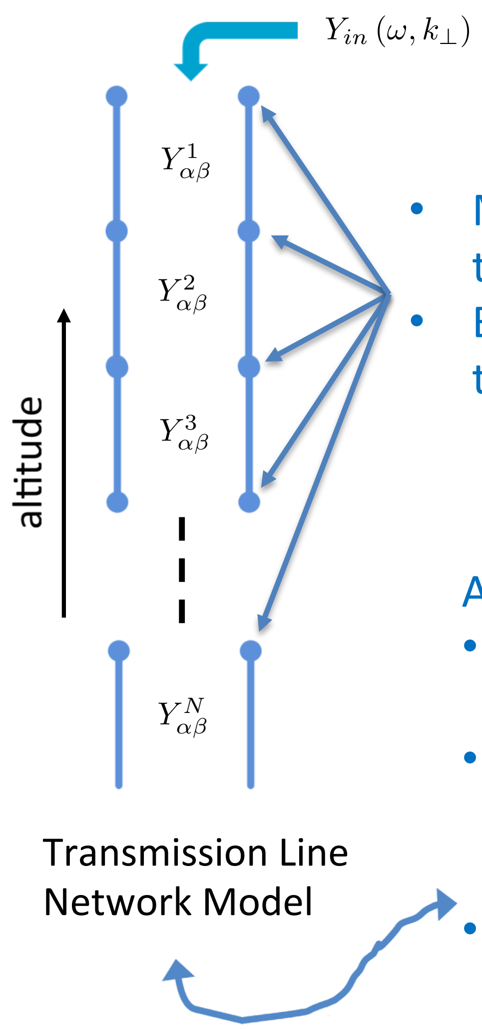
Exact Initial-Value Solution.

Stack Slabs and Use Two Wave Modes

Polarization Vectors



- **16 equations gives 8 potential waves.**
- **The X-mode, O-mode and Z-mode waves are too high in frequency.**
- **There is also a non-propagating pair.**
- **That leaves 4 waves.**
- **Only the Whistler and Alfven can travel more than 10 km.**
- **Whistler is cutoff in F region.**



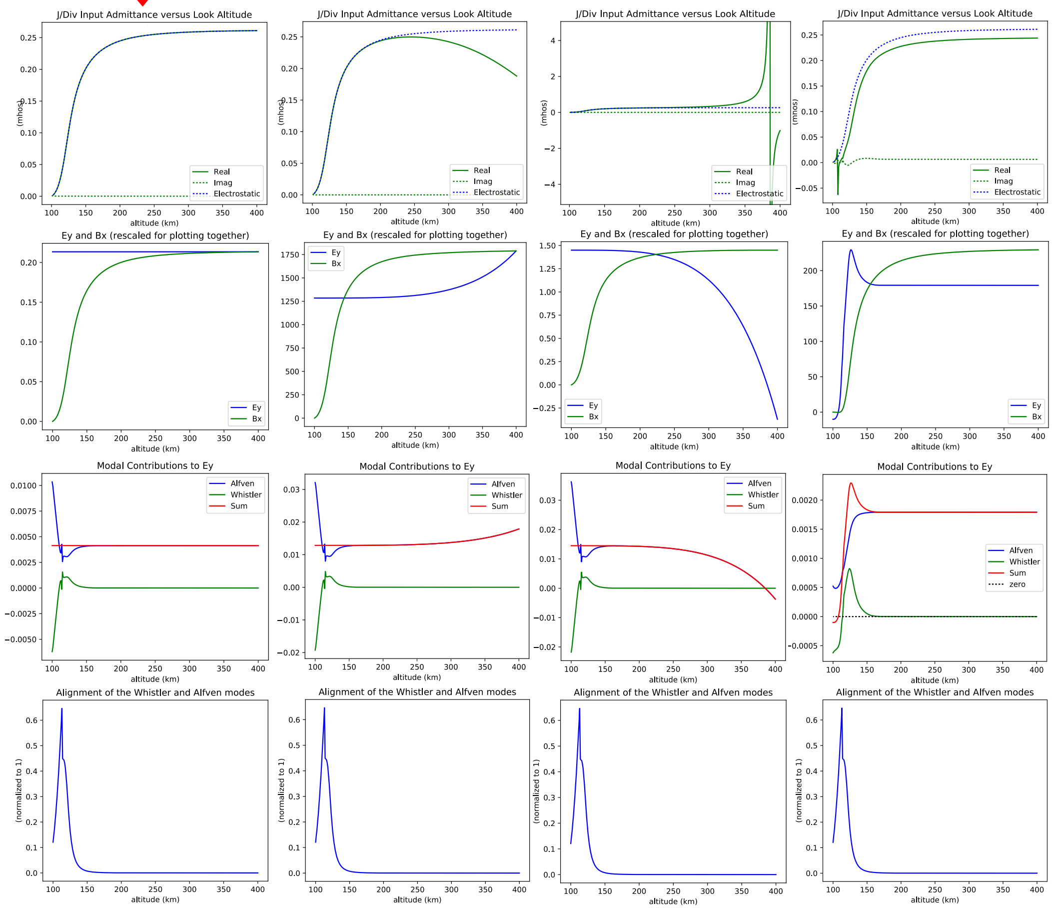
- Match across the boundaries.
- Bootstrap from the last section.

Approximations:

- Keep only the Alfven and Whistler terms.
- Usual interpretation of a propagating wave packet.
- Model ionosphere as stack of thin homogeneous slabs.

Validation and Three Types of Wavelike Effects

Exact reproduction of electrostatic theory



Ionospheric conductance versus altitude compared to electrostatic.

Mapping of E-field (blue) and B-field (green).

E-field contributions from Alfven (blue) and Whistler (green) modes.

Alignment of Alfven and Whistler modes. There is a near degeneracy!

Modify the waves to satisfy the conditions.

Decrease only the dissipation scale length.

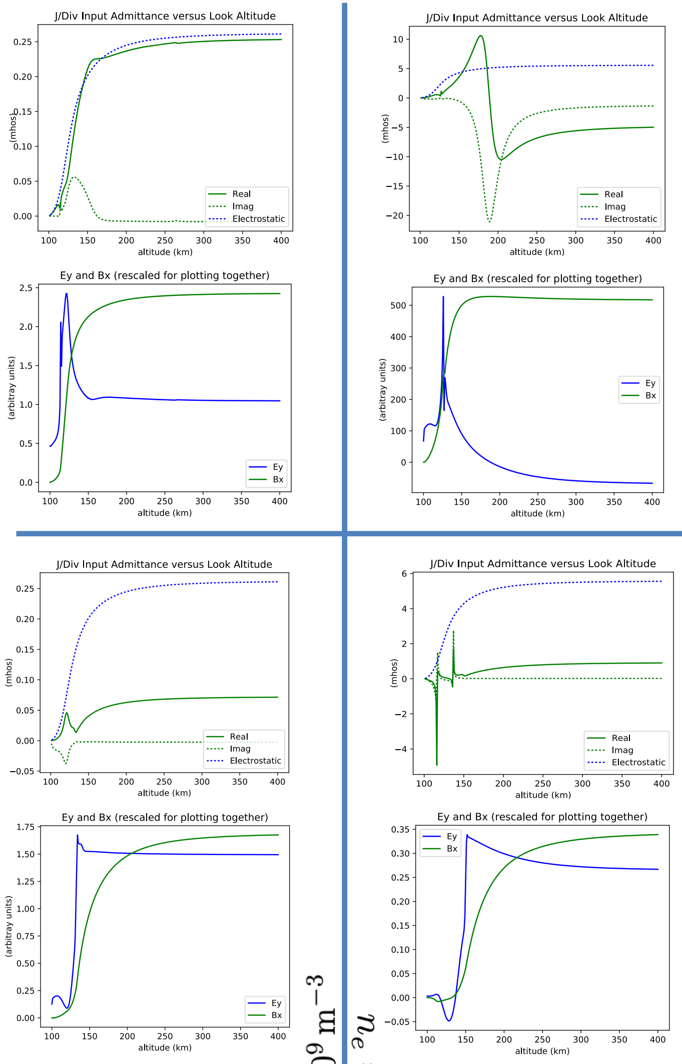
Decrease only the wavelength.

Make wavelengths unequal without decreasing.

$$\lambda_\perp = 100\,\mathrm{km}$$

$$n_e = 4.7 \times 10^9\,\mathrm{m}^{-3}$$

With the Real Waves get Unexpected, New Physical Predictions!



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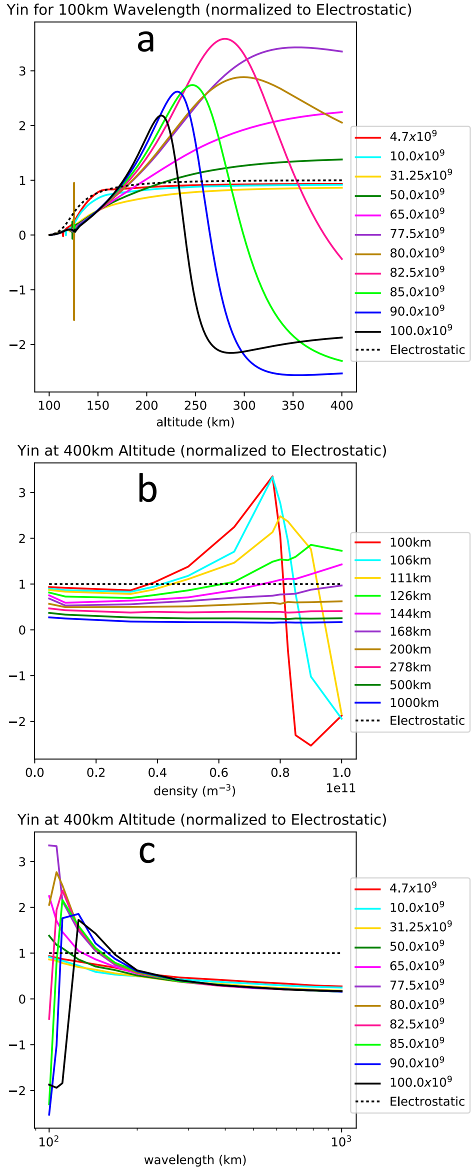
Ionospheric conductance versus altitude compared to electrostatic.

Mapping of E-field (blue) and B-field (green).

$n_e = 4.7 \times 10^9\,\mathrm{m}^{-3}$

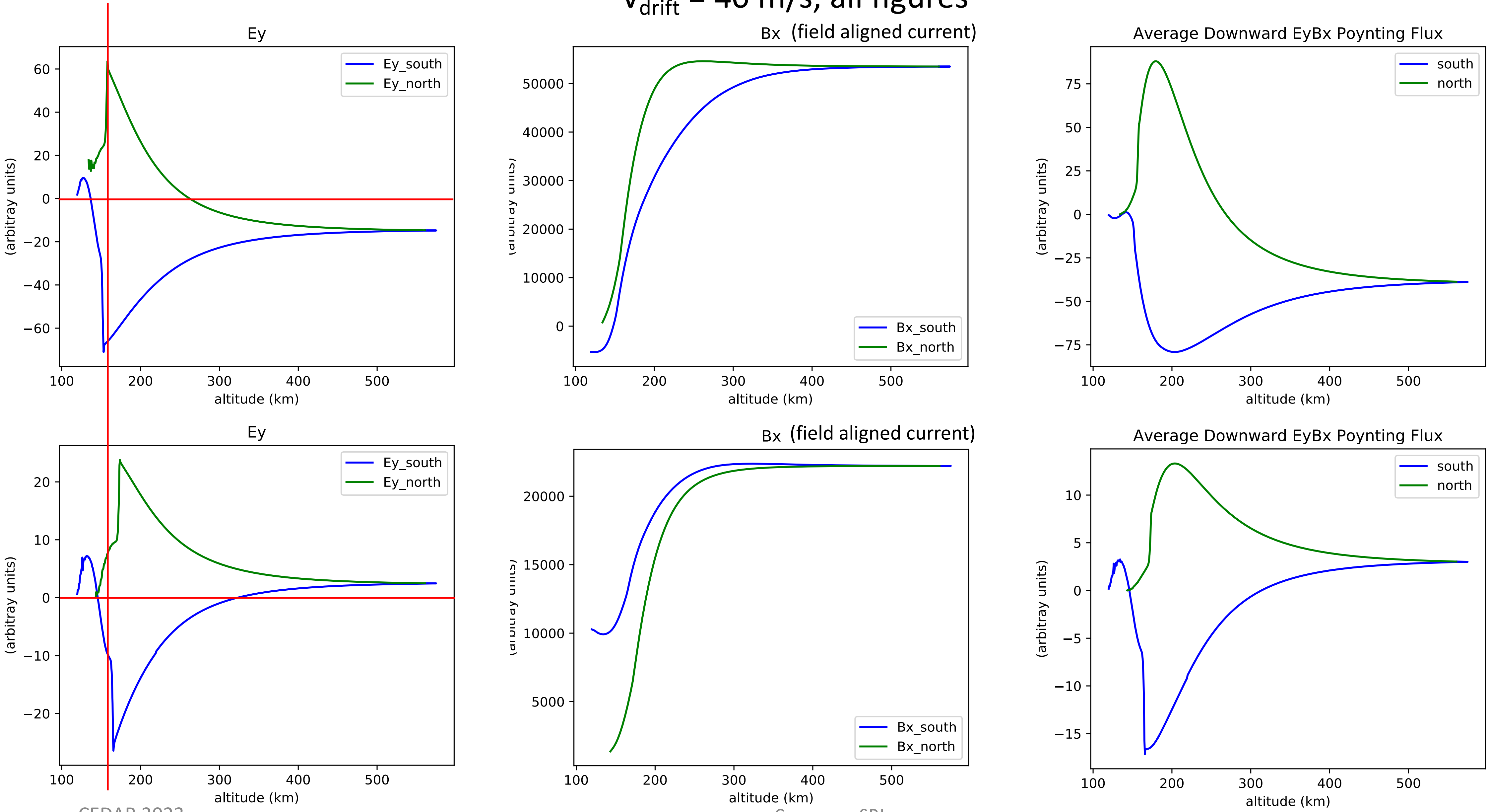
$n_e = 10^{11}\,\mathrm{m}^{-3}$



Ionospheric Conductance

Analyzing the Field Line from Tonga to ICON

V_{drift} = 40 m/s, all figures



1000km wavelength

2000km wavelength

Conclusions

- The model would predict electrostatic theory if the waves had the “right” properties.
- But the real, collisional ionospheric waves do not have these properties.
- The results are of a fundamental nature and affect many things.
- e.g.: (1) a resonance is possible between the *E* and *F* regions; (2) E field cutoff, greatly reducing Σ.
- This is a very standard physics calculation. It is just something that has been neglected.

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