Signal Statistics and Parameter Estimation

Ashton S. Reimer

University of Saskatchewan
and
SRI International

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Measurements and Physics

- Scientific instrument
- Rarely directly measure te physical parameters that we desire
- Derive parameters from Physics-based model (usually a forward model)
- Use fitting techniques that rely on data, model, and variance in data
- Feed other physics models with fitted parameters
- Are errors propagated properly? Need variance of fitter parameters

Variance of the data and parameters can be derived if we have knowledge of statistical properties of the measured signal.

Measurements vs. Estimates

Example, **measure** a voltage:

$$\tilde{V}_i(t)$$

Estimate an autocorrelation function

$$\hat{R}_i(\tau) = \tilde{V}_i^*(t)\,\tilde{V}_i(t+\tau)$$

- Derive an **estimate** from a **measurment**.
- Often these estimates are just called ‘‘data’’
- $\tilde{V}_i(t)$ has some statistical properties
- $\hat{R}_i(\tau)$ will have some statistical properties
- The properties of $\hat{R}_i(\tau)$ depend on the properties of $\tilde{V}_i(t)$

Fitting the data:

$$\chi^2 = \sum_{n=1}^{N} \frac{(d_n - m_n)^2}{\sigma_n^2}$$

for N data points, where d_n is the data values, m_n is the model values, and σ_n^2 is the variance of the data.

Danger:

* If you get the model wrong, fitted parameters are probably invalid
* If you get the variance wrong, fitted parameters are probably invalid

If model and variance are correct and fitted errors are derived properly, fitted errors will be very large if data is bad. **Goal is to propagate statistical uncertainty (variance) of measurements to fitted parameter errors.**

As an example: Look at radar measurements and estimated parameters.

SuperDARN radar in Saskatoon, Saskatchewan, Canada

AGU PUBLICATIONS

Radio Science

RESEARCH ARTICLE
10.1002/2016RS005975

On the statistics of SuperDARN autocorrelation function estimates

A. S. Reimer[1], G. C. Hussey[1], and S. R. Dueck[1]

[1]Department of Physics and Engineering Physics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada

Key Points:
- Statistical distributions and variance of SuperDARN autocorrelation function estimates are analytically derived
- Distributions and variance are verified via Monte Carlo simulations and experimental data
- Variance of autocorrelation function estimates are needed for correctly performing error-weighted fitting of SuperDARN data

Correspondence to:
A. S. Reimer,
ashton.reimer@usask.ca

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Abstract Time domain signal processing techniques are employed by the Super Dual Auroral Radar Network (SuperDARN) to obtain bulk measurements of the velocity and spectral width of *F* region ionospheric plasma irregularities. The measurements are obtained by fitting estimates of the mean autocorrelation function (ACF) of the radar target. To accurately and consistently extract target parameters from the mean unnormalized ACF, it is necessary to utilize error-weighted fitting algorithms with a weight given by the variance of the ACF. Currently implemented weights are ad hoc, and a detailed description of the statistical characterization of SuperDARN ACFs is needed. Following the discussions in Farley (1969) and Woodman and Hagfors (1969), which describe the variance for the mean normalized ACF used with incoherent scatter radars, we present analytic expressions for obtaining the variance of the real and imaginary components of the mean unnormalized SuperDARN ACF. These expressions are based on models by André et al. (1999) and Moorcroft (2004) of the voltage signal received by SuperDARN radars but may be used for other soft target radar systems. An algorithm for obtaining the variance of both the magnitude and phase of the mean ACF is also presented. The results of this study may be directly integrated into existing SuperDARN data analysis software and other pulse-Doppler radar systems that utilize estimates of the mean unnormalized ACF.

For SuperDARN:

$$R(\tau) = Pe^{-2\pi w_d \tau/\lambda} e^{j2\pi f_d \tau}$$

- $P \equiv$ the echo power
- $w_d \equiv$ the spectral width
- $f_d \equiv$ the Doppler frequency

For ISRs:

$$S(N_e, T_e, T_i, V_d) \rightarrow \mathsf{FFT} \rightarrow R(\tau)$$

- $N_e \equiv$ electron density
- $T_e \equiv$ electron temperature
- $T_i \equiv$ ion temperature
- $V_d \equiv$ the Doppler velocity

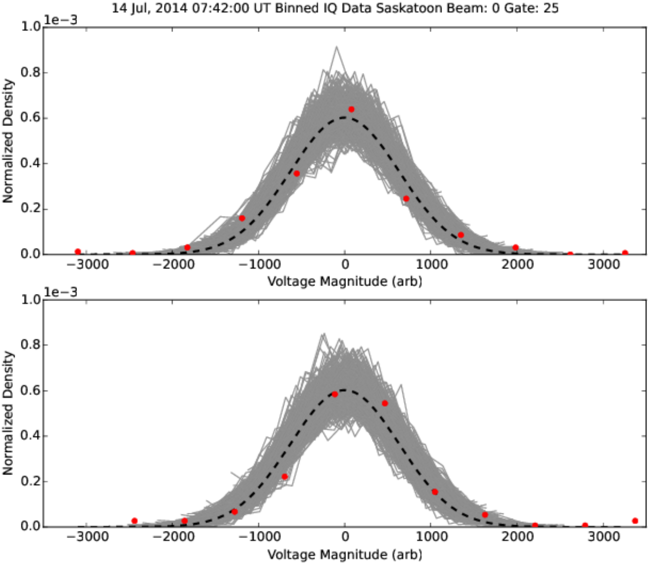
Measure complex voltage samples:

$$\tilde{V}_i(t)$$

Estimate an time integrated autocorrelation function (ACF)

$$\hat{R}(\tau) = \sum_{k=1}^{K} \tilde{V}_k^*(t)\,\tilde{V}_k(t+\tau)$$

If we know the statistics of the voltage samples, we can determine the statistics of the ACF estimates.


14 Jul, 2014 07:42:00 UT Binned IQ Data Saskatoon Beam: 0 Gate: 25
1e−3
Normalized Density
Voltage Magnitude (arb)
1e−3
Normalized Density
Voltage Magnitude (arb)

Voltages are zero-mean correlated Gaussian random variables.

With $\tilde{V}(t) = x_1 + ix_2$ and $\tilde{V}(t+\tau) = x_3 + ix_4$:

$$p(x_1, x_2, x_3, x_4) = \frac{1}{(2\pi)^2|\mathbf{C}|^{1/2}} \exp\left(-\frac{1}{2|\mathbf{C}|}\sum_{i,j=1}^{4}|C_{ij}|x_i x_j\right)$$

$$\mathbf{C} = \sigma^2 \begin{bmatrix} 1 & 0 & \rho_r & -\rho_i \\ 0 & 1 & \rho_i & \rho_r \\ \rho_r & \rho_i & 1 & 0 \\ -\rho_i & \rho_r & 0 & 1 \end{bmatrix}$$

* From Farley and Hagfors, unpublished ISR textbook.

Using the Central Limit Theorem:

$$p(x,y) = \frac{1}{2\pi\sigma_r\sigma_i\sqrt{1-\rho_{ri}^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_r^2\left(1-\rho_{ri}^2\right)} - \frac{(y-\bar{y})^2}{2\sigma_i^2\left(1-\rho_{ri}^2\right)} + \frac{\rho_{ri}(x-\bar{x})(y-\bar{y})}{\sigma_r\sigma_i\left(1-\rho_{ri}^2\right)}}$$

where

$$\rho_{ri} = 2\rho_r\rho_i$$

- x and y correspond to estimates of the real (r) and imaginary (i) components of a lag of the ACF at time τ
- bar notation denotes expected values
- ρs are normalized correlation coefficients
- σ^2s are variances

* From Reimer et. al. (2016) On the statistics of SuperDARN autocorrelation function estimates. Radio Science.

From Estimator theory: the variance of the ACF estimator ($\sigma^2_{r,i} =< \hat{R}^2_{r,i} > - < \hat{R}_{r,i} >^2$):

$$\sigma_r^2 = P^2 \left(\frac{1 - |\rho|^2}{2K} + \frac{\rho_r^2}{K} \right) \tag{1}$$

and

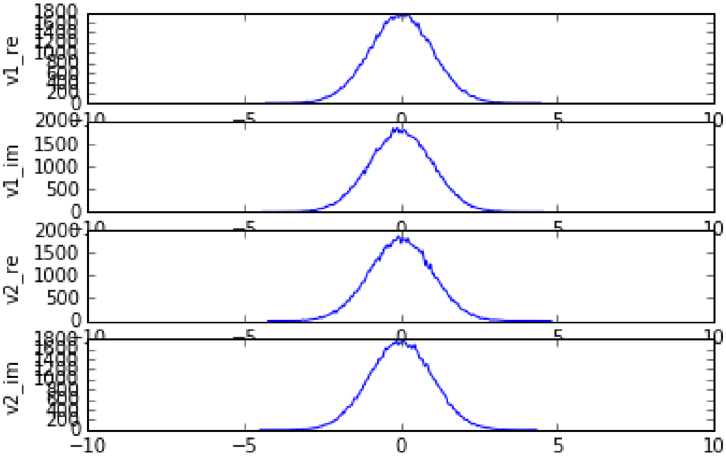
$$\sigma_i^2 = P^2 \left(\frac{1 - |\rho|^2}{2K} + \frac{\rho_i^2}{K} \right) \tag{2}$$

* From Reimer et. al. (2016) On the statistics of SuperDARN autocorrelation function estimates. Radio Science.

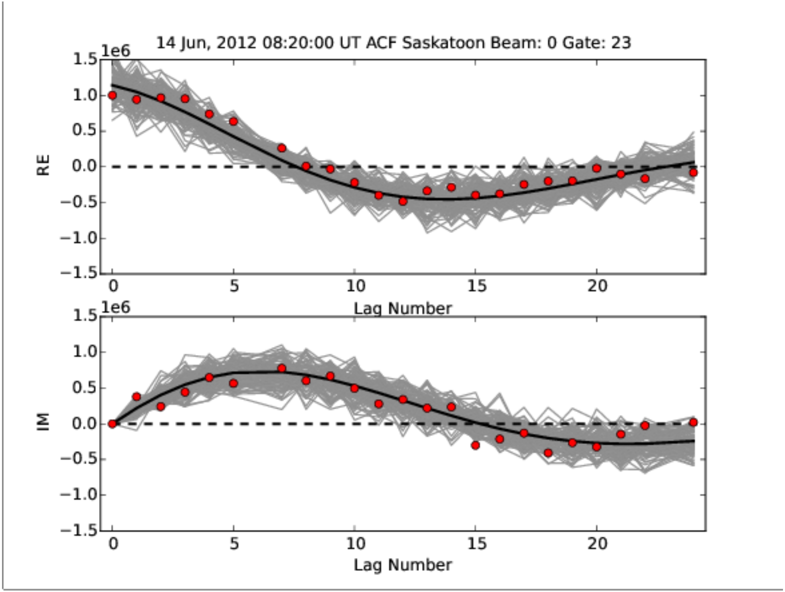
With knowledge of the probability distribution function of the voltage samples:

1. Sample the distribution to generate numeric ‘‘measurements’’
2. Determine estimates of the ACF
3. Repeat step 1 a large number of times to build up a distribution of the ACF estimates
4. Numerically estimate the variance from the ACF estimates

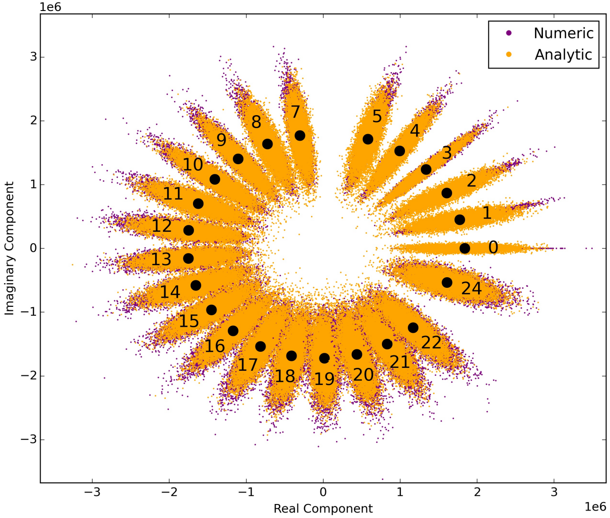
1. Sample the distribution to generate numeric ‘‘measurements’’. Distribution of these measurements:



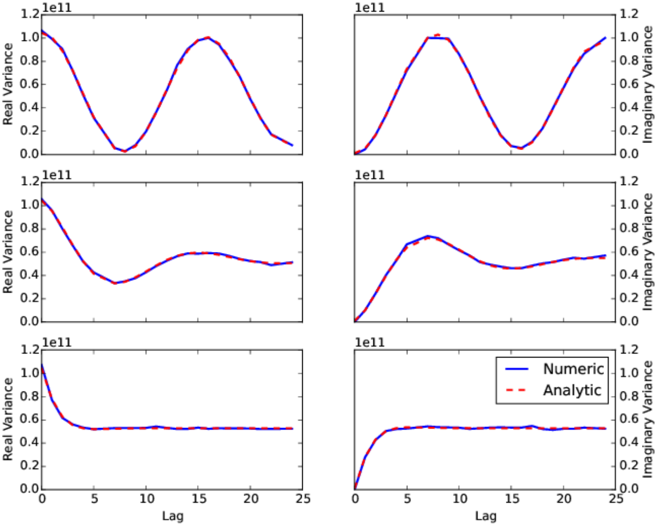
2. Determine estimates of the ACF:



3. Build up a distribution of the ACF estimates:



4. Variance of ACF estimates:

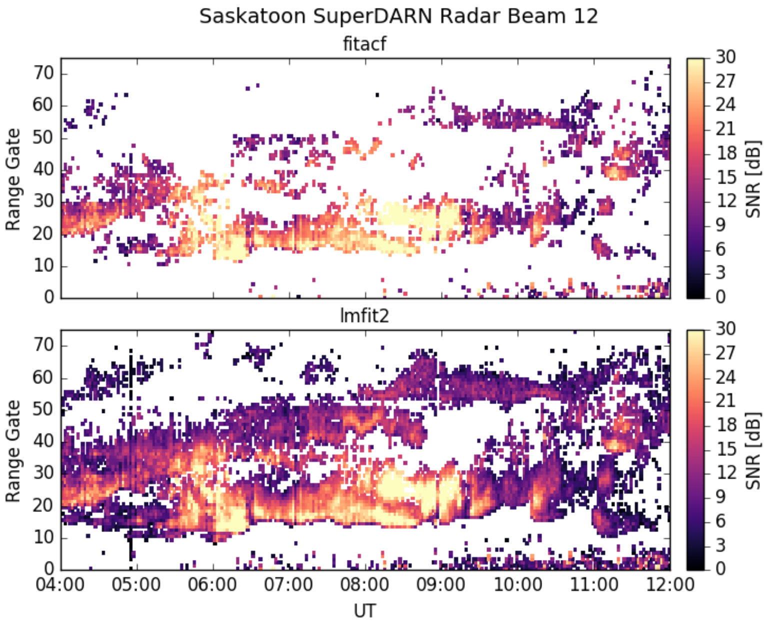


Summary:

* Determine the statistical properties of a measured signal
* Derive statistical properties of data derived from measured signal
* Propagate statistical uncertainty of fitted data to fitted physical parameters

This enables the production of self-consistent and reliable error bars for fitted parameters derived from instrument measurements!

Proof on the next 3 slides:


Saskatoon SuperDARN Radar Beam 12
fitacf
lmfit2
Range Gate
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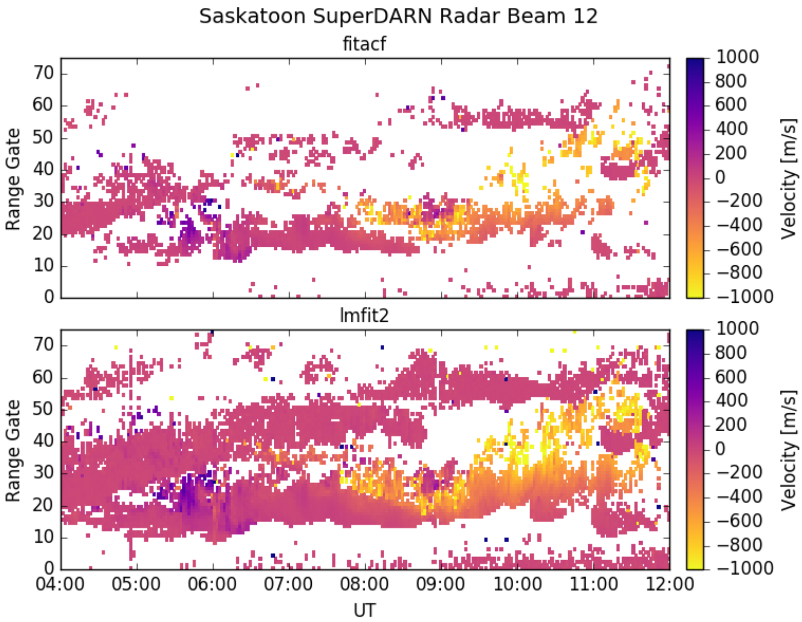
Saskatoon SuperDARN Radar Beam 12
fitacf
lmfit2
Range Gate
Velocity [m/s]
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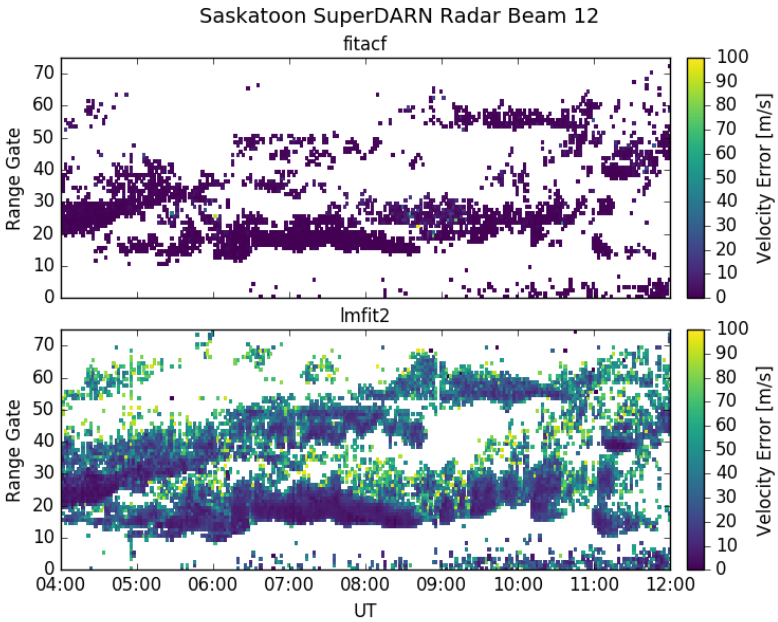
Saskatoon SuperDARN Radar Beam 12
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