# Two-dimensional Morlet wavelet transform and its application to extracting two-dimensional wave packets from lidar observations in Antarctica

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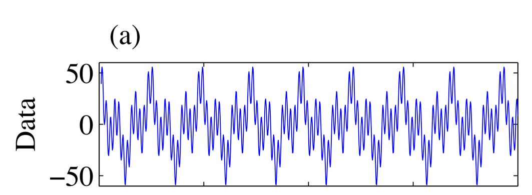
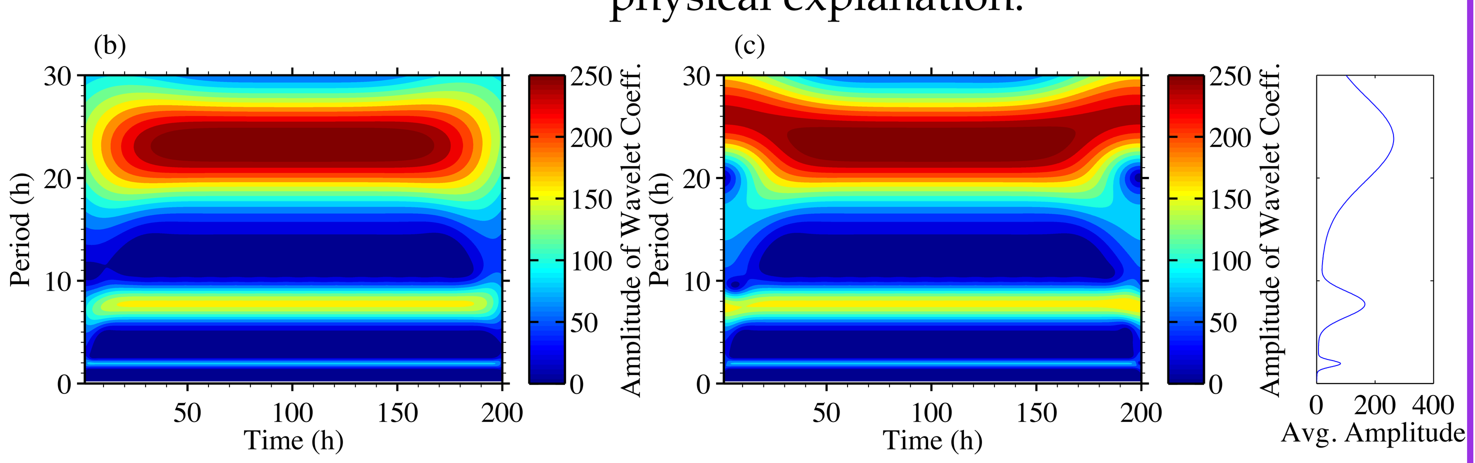
**Abstract** Corrections to the commonly used 1-D Morlet wavelet transform code by Torrence and Compo (1998) are provided to ensure that the power of the wavelet transform is unbiased. We then develop 2-D wavelet transform and reconstruction method to automatically extract 2-D gravity wave packets. By applying this new method to lidar data, we demonstrated the utility to study recently discovered persistent gravity waves in Antarctica. The results are in line with the conclusion in Chen et al. (2016) that inertia-gravity waves are persistent and dominant, and exhibit lifetimes of multiple days in winter. The variations in the extracted wave properties indicate a month-to-month variability.

## 1. Introduction

Wavelet transform is very attractive to spectral analysis and extraction of waves from atmospheric and space observational data. However, problems were found in the publicly available 1-D wavelet code provided by Torrence and Compo (1998), i.e., wavelet power spectra are distorted or biased in favor of large scales or low frequencies (see plots below). Consequently, a correction is needed, which should also be supported by a full mathematical derivation and a physical explanation.





**Bias in the publicly available 1-D wavelet code: (a) Input time series of sine waves of three different periods (2 h, 8 h, 24 h) with same amplitude of 20 K. (b) Results of the Torrence and Compo (1998) code. (c) Results of the MATLAB wavelet toolbox cwtft function.**

Many remote sensing instruments deliver two-dimensional (2-D) data. extracting intermittent/localized two-dimensional wave packets is still a common technical challenge in analyzing atmospheric and space data. However, no 2-D Morlet wavelet code suitable for geophysical applications is publicly available

## 2. Correction for commonly used 1-D wavelet power spectrum

1-D wavelet transform

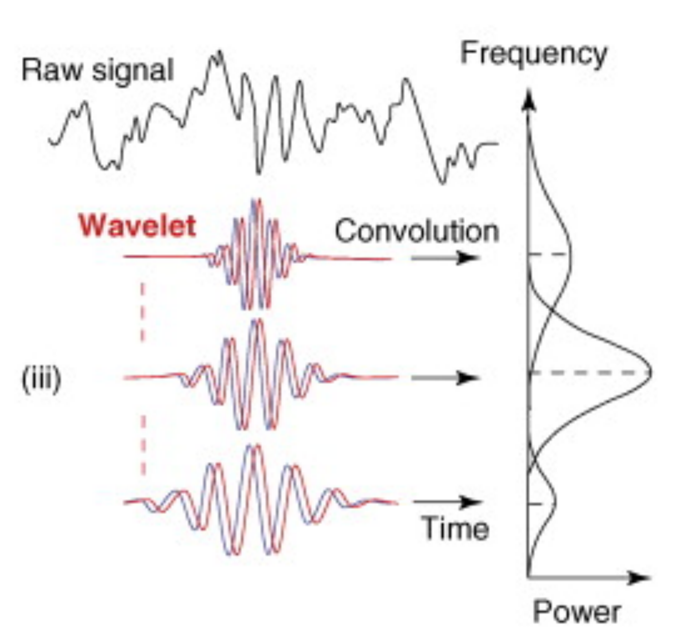
$$W_{f\psi}(s,t)=\int_{-\infty}^{+\infty} f(t')\frac{1}{\sqrt{s}}\psi^*\left(\frac{t'-t}{s}\right)dt'$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\sqrt{s}\left[\hat{\psi}(s\omega)\right]^* e^{i\omega t}d\omega$$

Morlet wavelet $\psi(t)=e^{i\omega_0 t}e^{-\frac{t^2}{2}}$

Energy conservation in the transform

$$\int_{-\infty}^{+\infty}|f(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\hat{f}(\omega)\right|^2d\omega$$



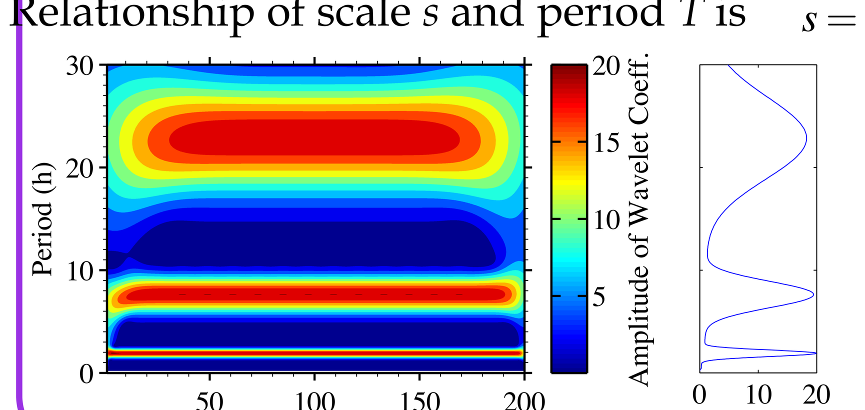


From that, we derive our **corrected wavelet power spectrum**

$$P_f(s,t)=\frac{1}{C_\psi'^2}\frac{\left|W_{f\psi}(s,t)\right|^2}{s}$$

Substituting $f$(t) with a cosine function of amplitude $A$, we set the wavelet power at equals the signal's mean power (averaged squared amplitude 0.5A²). We obtain $C_\psi'=\sqrt{\pi}$

Relationship of scale $s$ and period $T$ is $s=\omega_0/\omega_n=\omega_0 T/2\pi$



**Results of our corrected 1-D wavelet power spectrum give correct and unbiased absolute amplitudes. Best suitable for automated wave packet extraction**

## 3. Two-dimensional Morlet wavelet power spectrum

The 1-D wavelet can be extended to two or more dimensions by rotation, dilation and translation.

$$W_{f\psi}(s,\theta,\vec{t})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\vec{t}')\frac{1}{s}\psi^*\left(\mathbf{\Omega}_\theta^{-1}\frac{\vec{t}'-\vec{t}}{s}\right)d\vec{t}'$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\vec{\omega})s\left[\psi(s\mathbf{\Omega}_\theta\vec{\omega})\right]^*e^{i\vec{\omega}\cdot\vec{t}}d\vec{\omega}$$

2-D Morlet wavelet $\psi(\vec{t})=e^{i\vec{\omega}_0\cdot\vec{t}}e^{-\frac{|\vec{t}|^2}{2}}$

$$\mathbf{\Omega}_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

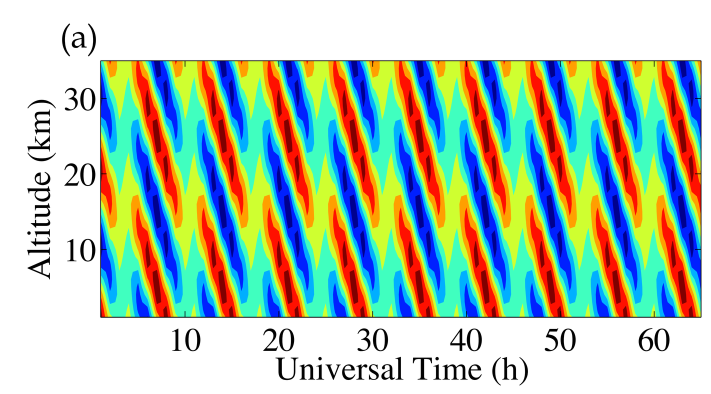
We derive the energy conservation in the transform

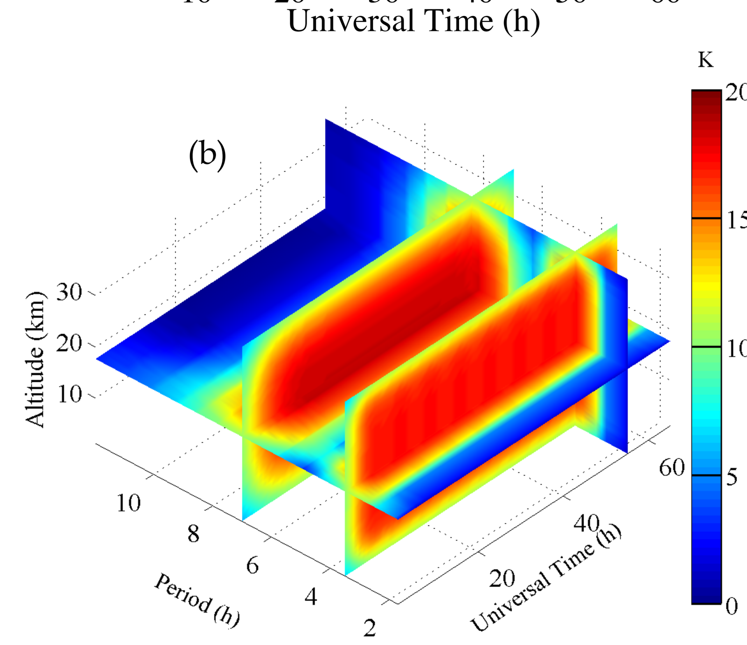
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|f(\vec{t})\right|^2d\vec{t}=\frac{1}{C_\psi}\int_0^{2\pi}\int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left|W_{f\psi}(s,\theta,\vec{t})\right|^2\frac{1}{s^2}d\vec{t}\frac{ds}{s}d\theta$$

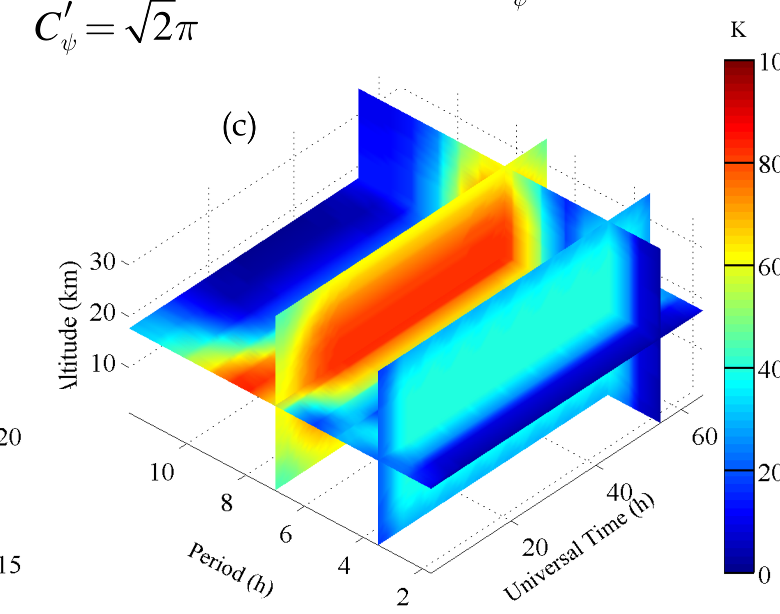
Our corrected 2-D wavelet power spectrum is

$$P_f(s,\theta,\vec{t})=\frac{1}{C_\psi'^2}\frac{\left|W_{f\psi}(s,\theta,\vec{t})\right|^2}{s^2}$$

$$C_\psi'=\sqrt{2\pi}$$







**(a) Input 2-D plane-sine waves of two different periods (3.5 h and 7 h) with same $\lambda_z$ of 20 km, same amplitude of 20 K. (b) Unbiased results of our corrected 2-D wavelet power spectrum at $\lambda_z$=20 km. (c) Biased results of the MATLAB wavelet cwtft2 function at $\lambda_z$=20 km.**

Relationship of scale $s$ and angle $\theta$ with $\lambda_z$ and $T$ is $T=\frac{2\pi s}{\omega_0\cos\theta}$ $\lambda_z=-\frac{2\pi s}{\omega_0\sin\theta}$

## 4. Two-dimensional wavelet reconstruction

2-D wavelet inverse transform is

$$f(\vec{t})=\frac{1}{C_\delta}\int_0^{\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}s^{-3}\Re\left[W_{f\psi}(s,\theta,\vec{u})\right]\frac{1}{s}\delta\left(\mathbf{\Omega}_\theta^{-1}\frac{\vec{t}-\vec{u}}{s}\right)d\vec{u}\,ds\,d\theta$$

$$=\frac{1}{C_\delta}\int_0^{\pi}\int_0^{+\infty}\Re\left[W_{f\psi}(s,\theta,\vec{t})\right]\frac{1}{s}\frac{ds}{s}d\theta$$

with $C_\delta=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\left|\hat{\psi}(\omega,k)\right|}{2\sqrt{\omega^2+k^2}}d\omega\,dk<+\infty$

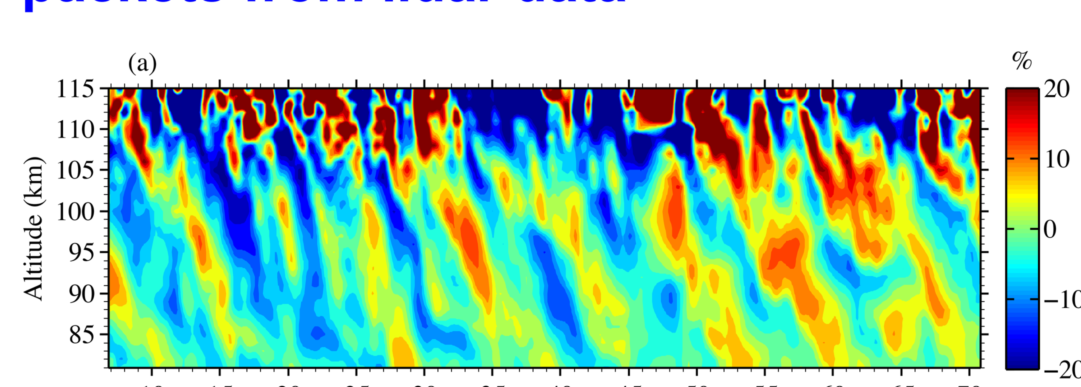
Discretized version

$$f[n_1,n_2]=\frac{1}{C_\delta}\sum_{l=0}^{L}\sum_{j=0}^{J}\frac{\Re\left\{W_{f\psi}[j,l,n_1,n_2]\right\}}{2^{\frac{j}{J}}s_0}\cdot\frac{\pi\ln 2}{JL}$$

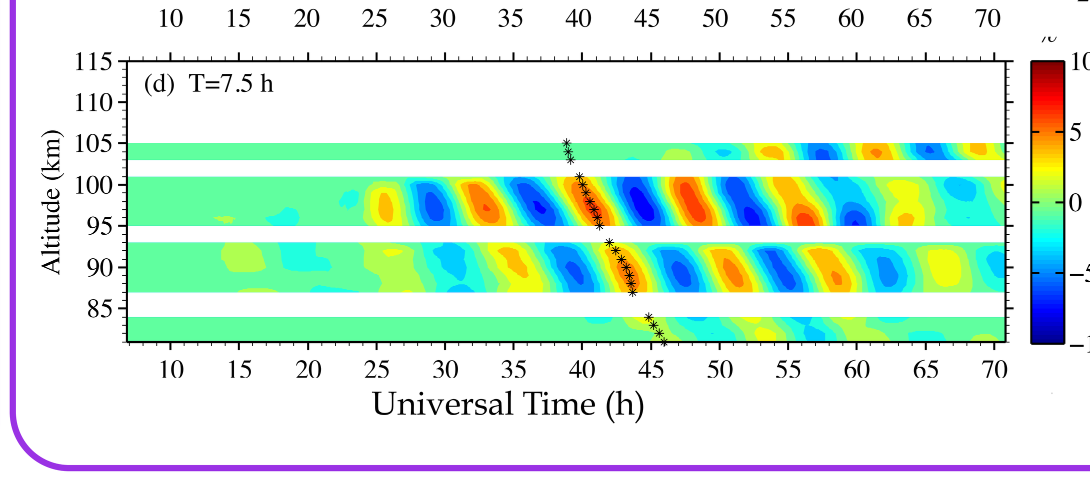
where

$$C_\delta=\sum_{l=0}^{L}\sum_{j=0}^{J}\Re\left[\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}2\pi e^{-\frac{\left(\frac{2^{\frac{j}{J}}s_0 2\pi k_1}{N_1\Delta t}-\omega_0\cos\left(\frac{l\pi}{L}\right)\right)^2+\left(\frac{2^{\frac{j}{J}}s_0 2\pi k_2}{N_2\Delta t}+\omega_0\sin\left(\frac{l\pi}{L}\right)\right)^2}{2}}\right]\cdot\frac{\pi\ln 2}{JL}$$

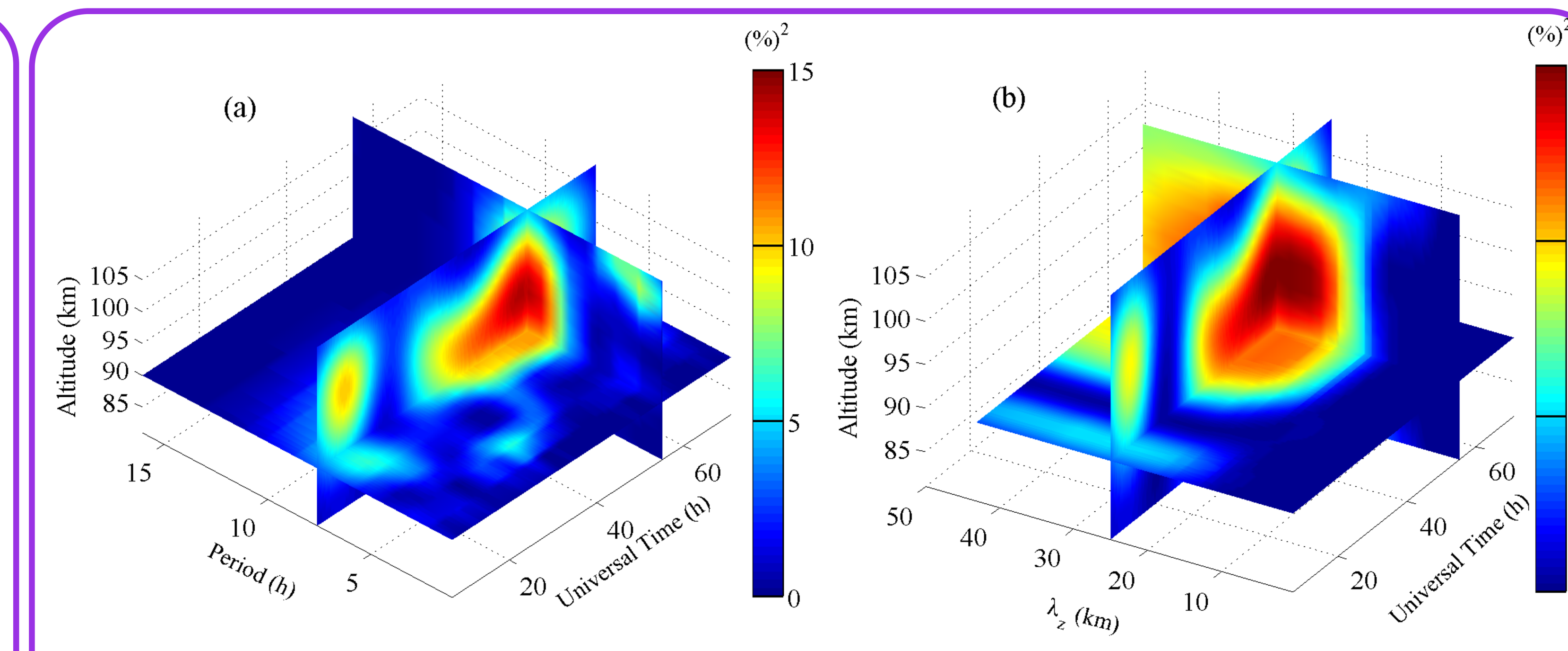
## 5. Applications to extracting two-dimensional wave packets from lidar data

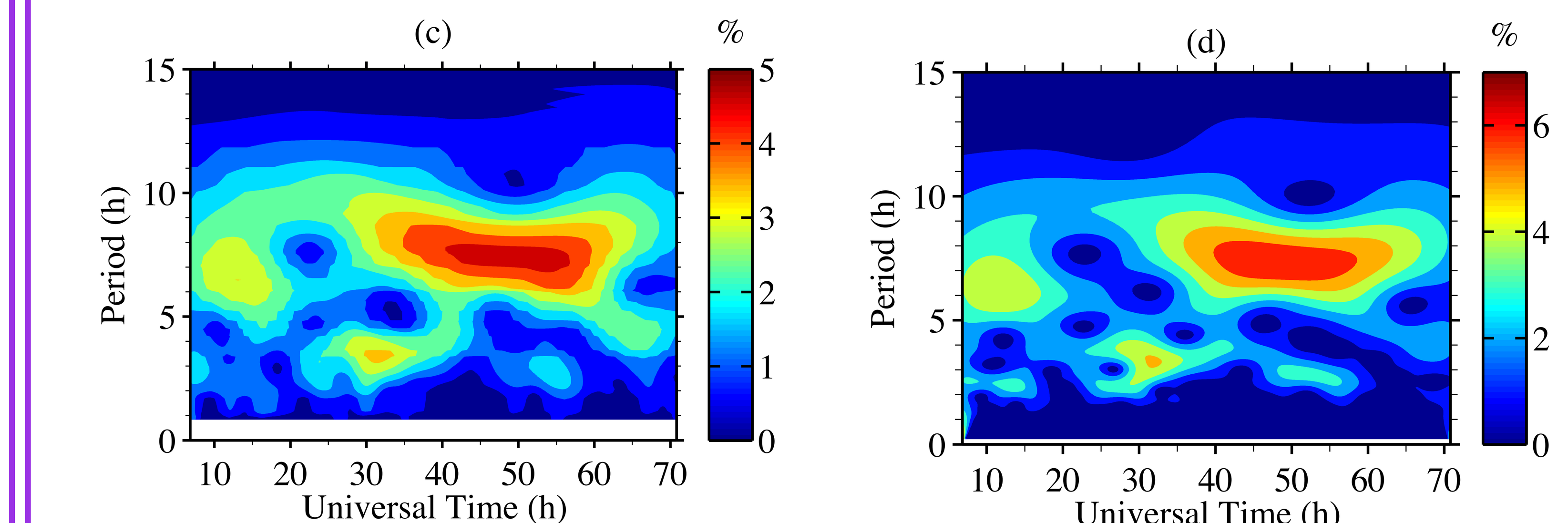


**Relative Temperature Perturbations on 28–30 June 2014 (upper left).**

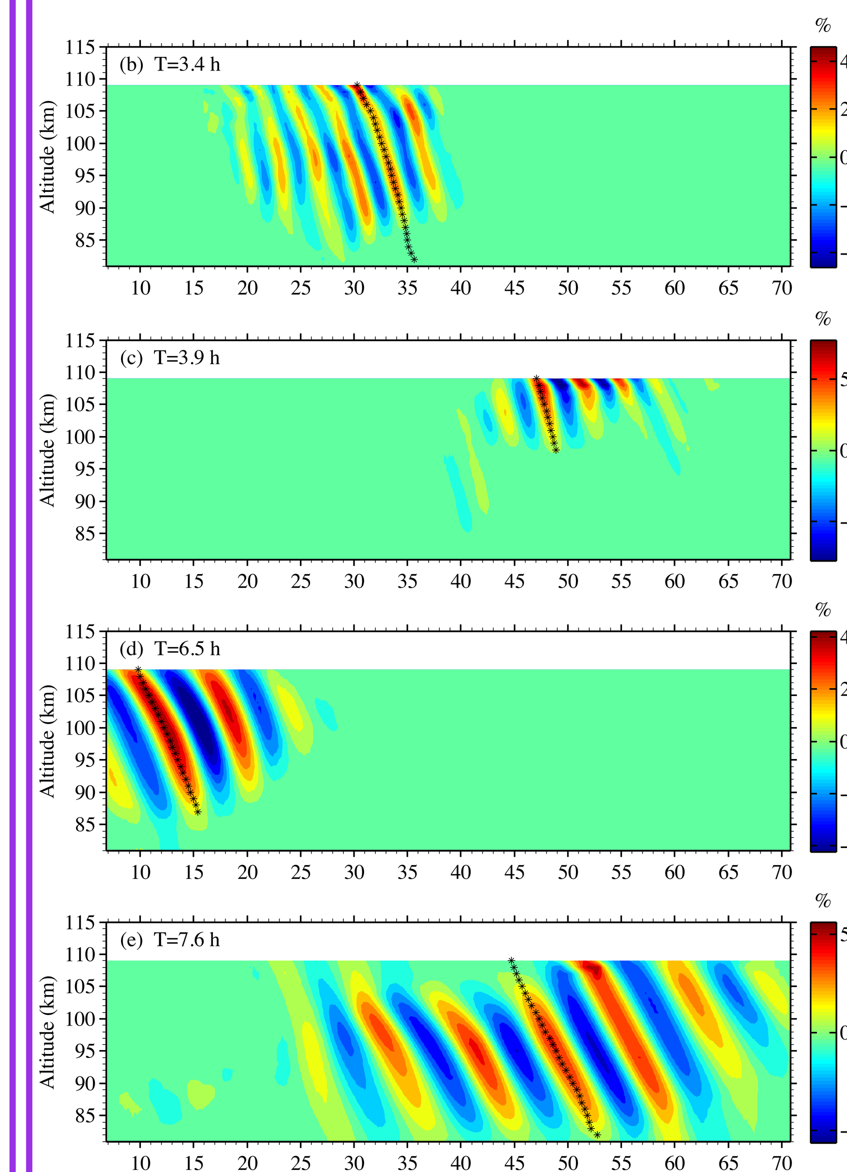


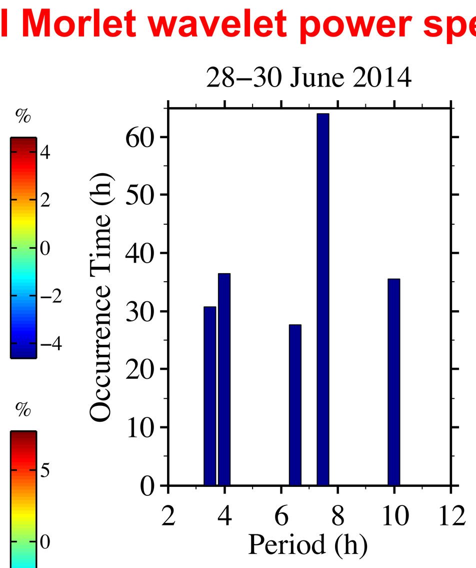
**Chen et al. (2016) used 1-D wavelet to extract gravity wave packets (bottom left). However, discerning each wave feature across the space domains had to be performed manually, which also left gaps.**



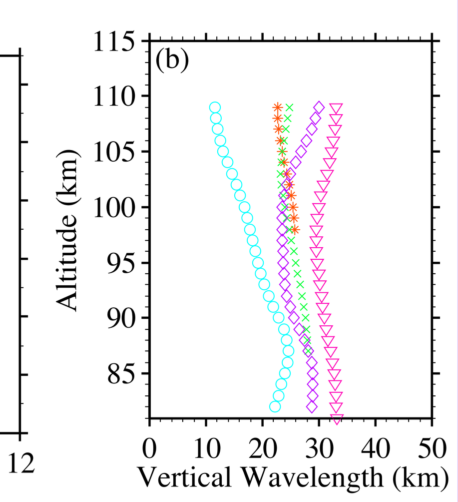


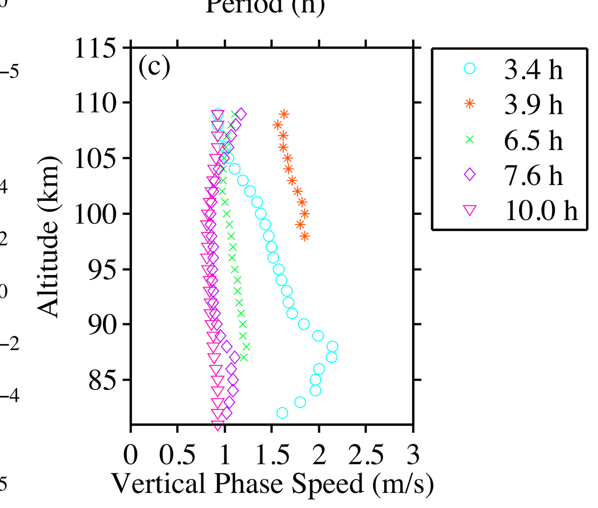
**2-D Morlet wavelet power spectrum at (a) λz = 25 km and (b) T = 7.6 h. (c) Two-dimensional top view of (a) showing how the wave period changes with time at z = 90 km and λz = 25 km. (d) One-dimensional Morlet wavelet power spectrum at z = 90 km.**



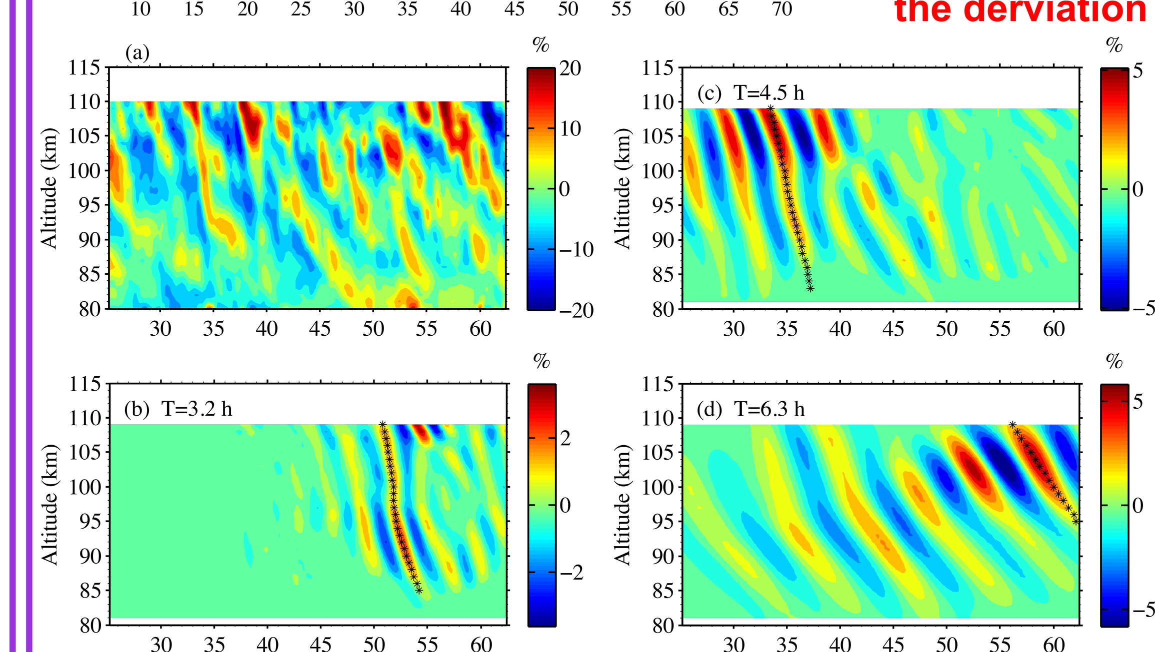








**2-D wave packets were automatically and truthfully recognized and reconstructed. The phases of the wavelet transform were directly used for the derviation of $\lambda_z$ and $c_z$.**



**Relative Temperature Perturbations 23–24 May 2014 and reconstructed 2-D wave packets.**

The phase speed of the 3.2-h wave exhibits a large variation with altitude, which could be the result of a more variable background wind condition.

## 6. Conclusions and outlook

We have developed a 2-D Morlet wavelet transform methodology for automatic extraction of quasi-monochromatic 2-D wave packets from 2-D datasets. This methodology can be used to analyzing large amounts of atmospheric and space science data. Furthermore, by implementing an isotropic wavelet function such as the Halo wavelet this method can be used to identify and extract ring structures in concentric gravity waves

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