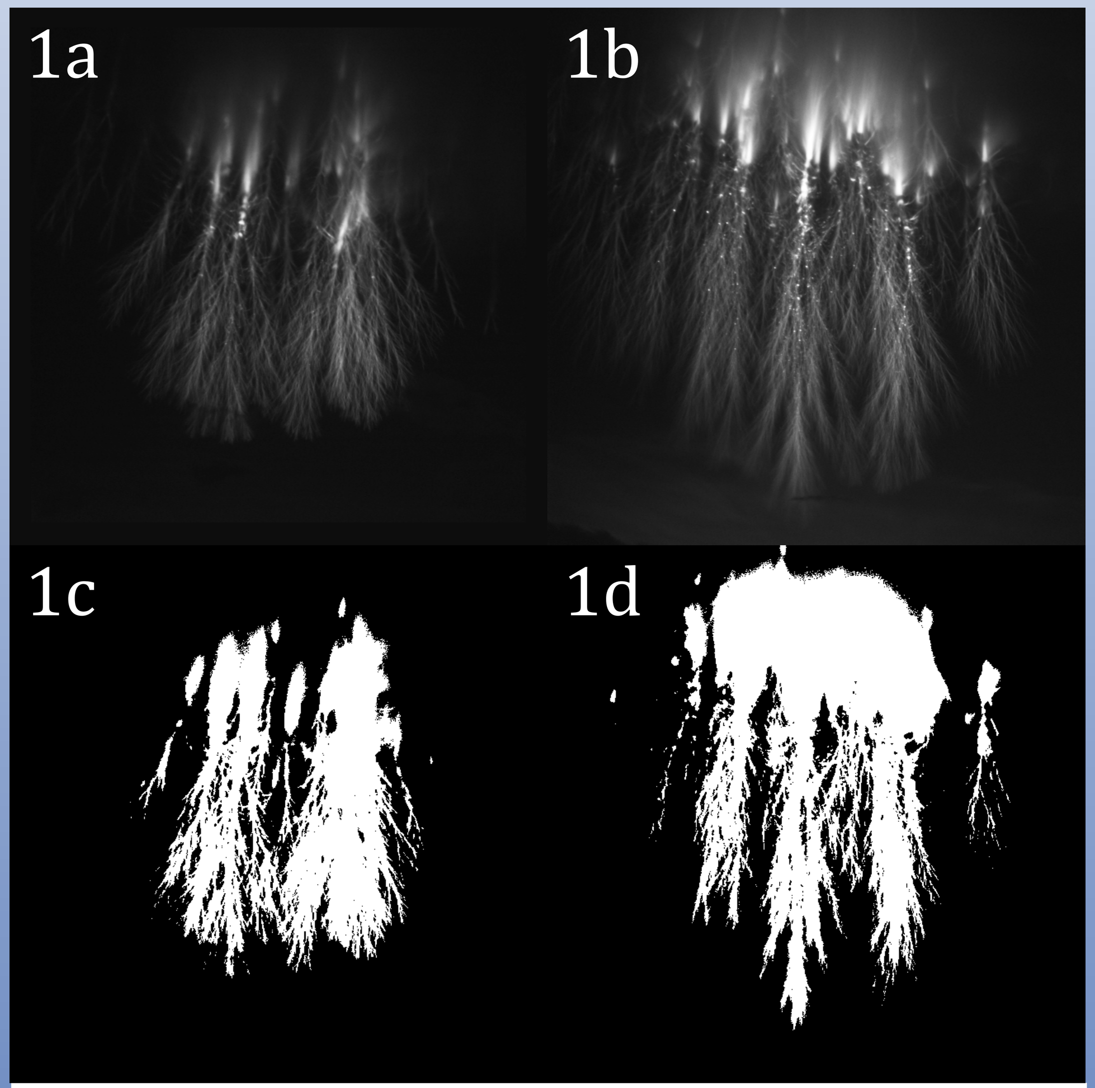
Using Multifractal Analysis to Advance Fractal Modeling of Sprites

P.A. Gough (Payson.Gough@UNH.edu), Dr. Ningyu Liu (Ningyu.Liu@UNH.edu)
Department of Physics and Astronomy, University of New Hampshire

This research was supported in part by NSF Grant AGS-2247153 and AFOSR Award FA9550-18-1-0358 to the University of New Hampshire.

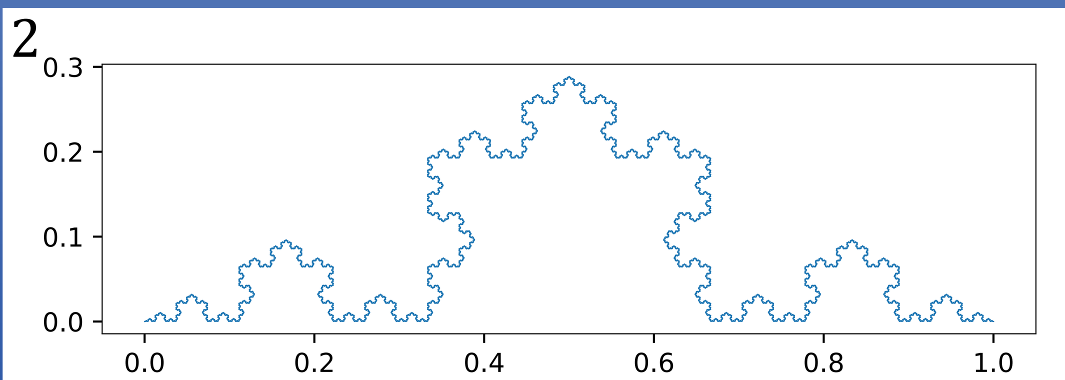
Abstract

Sprites are large streamer trees that propagate down from the ionospheric boundary. Thought to be scaled versions of streamer coronae, they are thus of wider interest to the study of lightning and spark discharges. Fractal modeling is a technique to simulate the large-scale behavior of dielectric breakdown, which has seen some success at simulating sprites. This poster presents work to improve upon existing fractal models and an analysis of the results using multifractal analysis.



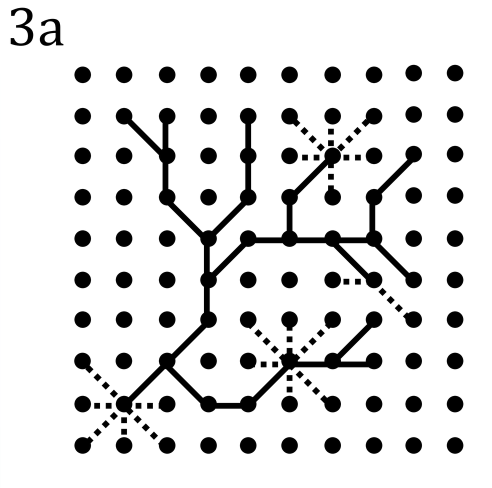
Sprites

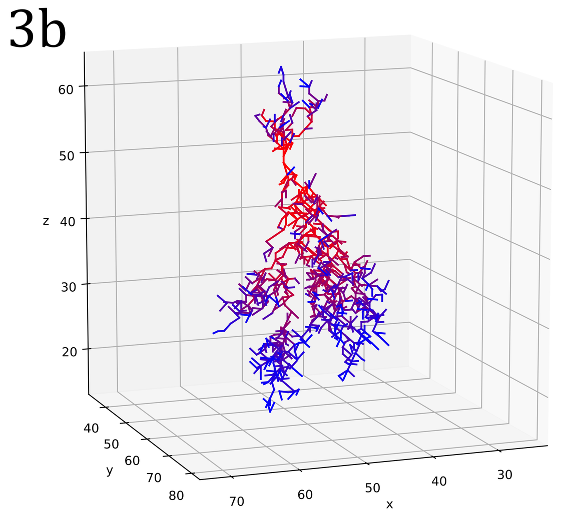
Figures 1a and 1b are high speed images of sprites, a form of high altitude lightning. Figures 1c and 1d show the same images filtered such that all pixels are either black or white, based upon a pixel brightness threshold. This method looses a lot of detail, but it is okay for basic analysis. We will be investigating the effects of other image processing techniques in the future.



Fractals

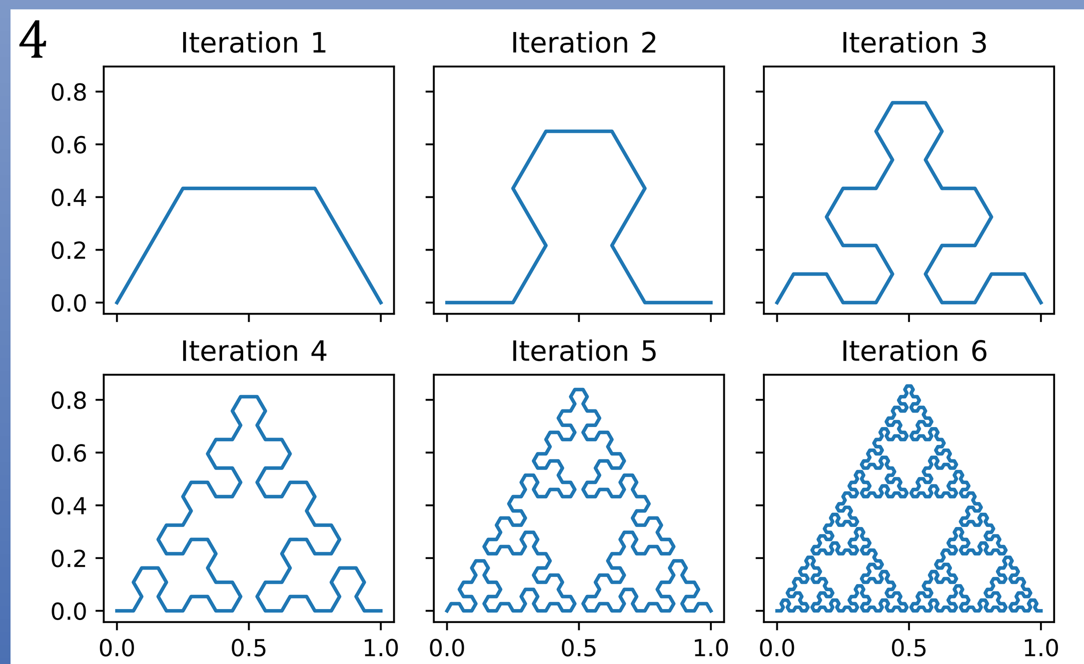
Figure 2 depicts the von Koch Curve, a famous examples of a man-made fractal. Even at the smallest scales, fractals contain infinite detail and are never smooth. The von Koch curve is generated iteratively, it begins life as a unit line segment, but an infinite series of perturbations transform it into a beautiful craggy curve. Since each perturbation makes the curve slightly longer, the end results is an infinitely long curve contained in a finite space. Topologically, the von Koch curve is still a line, however in another sense it has transcended its line-hood, but not yet achieved plane-hood. This can be stated more precisely in the language of fractal dimension.





Fractal Modeling (Sprite Simulation Technique)

Sprites can be difficult to model since they occur over a large range of scales. One approach, fractal modeling, eschews the fine scale mechanics in order to focus on the large scale dynamics (Niemeyer). The discharge tree is modeled as a series of links between grid points, as depicted in figure 3a (adapted from Pakso 2000). A three-dimensional grid (figure 3b) is usually preferable (Pasko 2002, Riousset 2007). At each stage of simulation, a list of all possible links between a point on the tree and a neighboring point off the tree is compiled. This has been depicted for four selected points in figure 3a. Each link is assigned a weight based on the voltage across it, and one is randomly selected and added to the tree. The electric potential is then recalculated at each grid point using the successive over-relaxation method. Simulation continues until the border is reached or the field is so reduced that no more links can be added to the tree. Figure 3b is a streamer tree modeled in this way. A few more modifications remain to be added to the code before it will be a true simulation of sprites. Namely realistic space charge distributions and an altitude dependent critical electric field. Red links were added first, blue last.

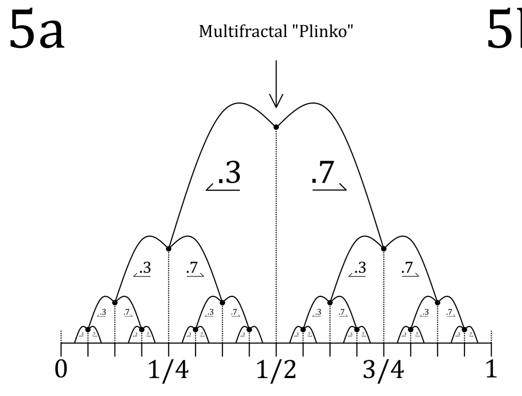


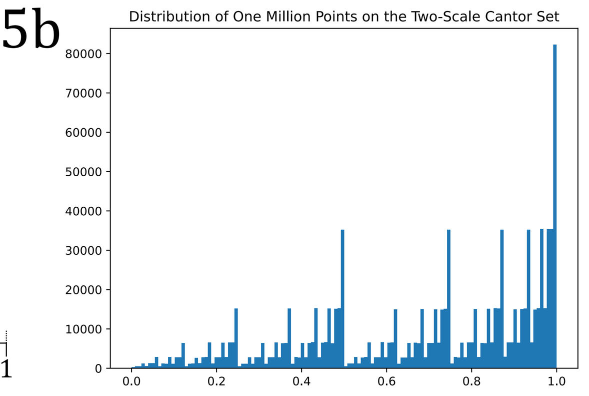
Fractal Dimension

Figure 4 depicts the first six iterations of the arrow head method of generating the Sierpinski triangle. You may be familir with other methods of generating this fractal, but this is the simplest to code and most apt for explaining the concept of fractal dimension. Compare the Sierpinki triangle to the von Koch curve. Both are infinitely long lines (at least after infinite iterations), but one might say the Sierpinki triangle is "denser", "crinklier", or "more plane-filling". Let us talk instead of the fractal dimension. For fractals like these, where we know the construction technique, the fractal dimension, D, is given by

$$D = \frac{\log(\text{number of copes})}{\log(\text{scale of copies})}$$

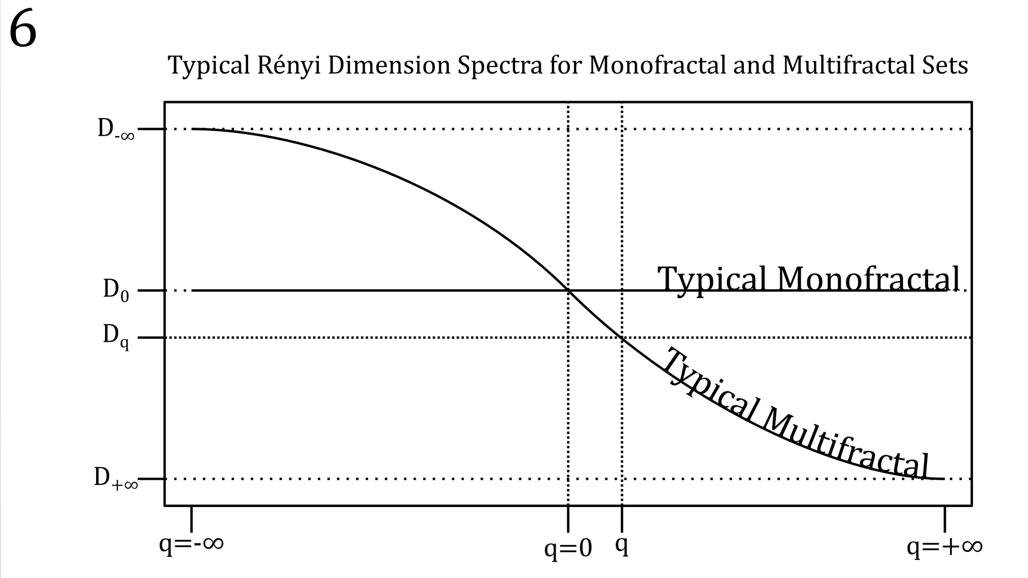
For the Sierpinski Triangle D = -log(3)/log(1/2)~1.58, for the von Koch curve D = -log(4)/log(1/3) ~ 1.26, which confirms our intuition about the relative "crinkliness" of the two curves. This method does not work for naturally occurring fractals such as sprites. Fortunately a veritable menagerie of methods have been invented to estimate their fractal dimension. Variation in the results of these methods leads us directly to the study of multifractals (Kisner)





Multifractals

The two-scale Cantor set is a simple example of a multifractal, which may be defined as a fractal with a nonuniform "mass" distribution. Figure 5a shows the "plinko" method of generating it. Figure 5b shows the final "mass" distribution. As a multifractal, it can't be characterized by a single fractal dimension, but rather by a generalized fractal dimension spectrum (Halsey).



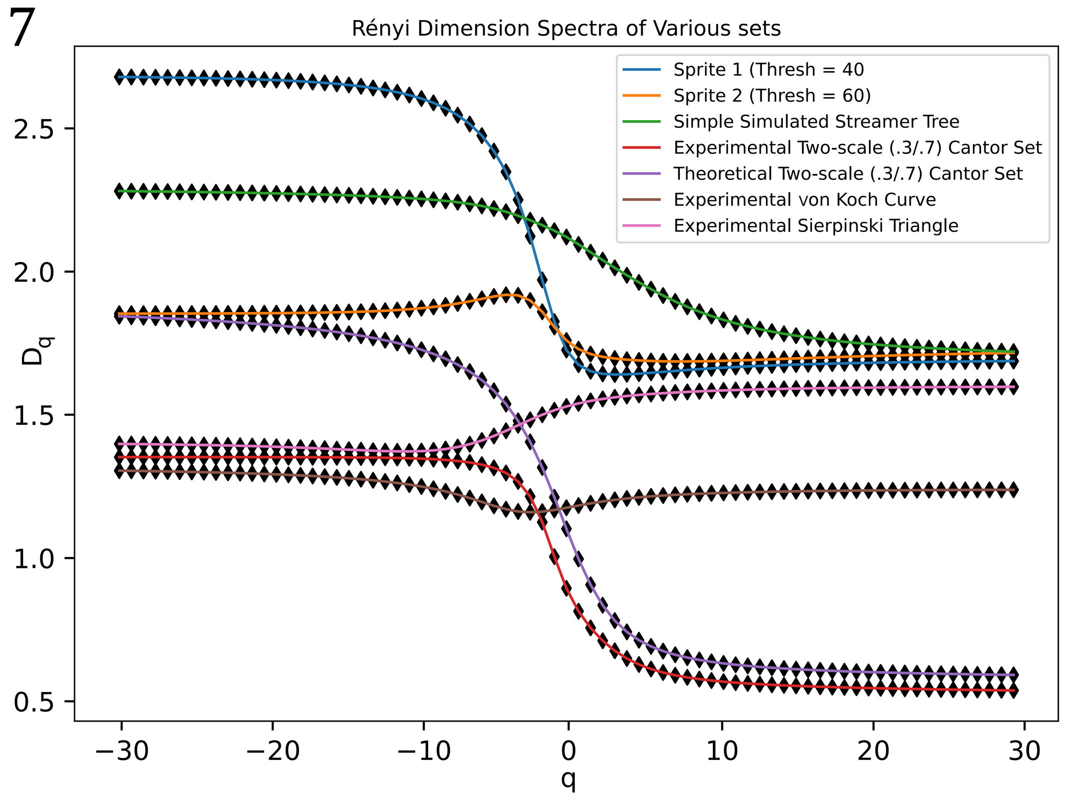
Generalized Dimension

As previously stated, there are many ways to estimate fractal dimension. Many methods differ in the way they are sensitive to the distribution of "mass" across the fractal. Such definitions can be considered special instances of a generalized fractal dimension D_q which is a function of the parameter q. For monofractals, such as the Sierpinski triangle or the von Koch curve, D_q should be single valued. For multifractals such as the two-scale Cantor set, it should be a strictly decreasing "S-curve", as depicted in figure 6. The simplest method of calculation is an extension of the box counting method, in which a grid is placed over the fractal. The boxes are of size r and P_i is the fraction of the number of points that fall within the i-th box. Then, the generalized fractal dimension is given by

$$D_q = \frac{1}{q-1} \lim_{r \to 0} \frac{\log(\sum_i P_i^q)}{\log(r)}$$

When D_q is plotted against q it is called the generalized dimension spectrum or Rényi dimension spectrum (Theiler). In practice, the box counting method is not very reliable, especially for negative values of q, because boxes that barely intersect with the fractal will be overcounted (Lopes). It was used to calculate the theoretical (exact) Rényi dimension spectrum for the two-scale Cantor set shown in figure 7. However, all the other curves used a more reliable albeit imperfect method called the sandbox method (Tél). The details of the sandbox method are beyond this presentation, but it suffices to say it uses concentriccircles rather than a square grid to cover the fractal.

The Rényi dimension spectrum can be transformed into the singularity spectrum f(α), which is generally considered to be preferable, because it has finite support and is more intuitive. f(α) is the fractal dimension of the distrubtion of subsets with with local dimension α. However, the transformation itself can introduce error. Chhabra proposed a direct method for calculating f(α), but it works poorly for negative values of q. Since I haven't found a reliable method of calculating it, I have chosen to use the Rényi dimension spectrum for this poster since it is more than sufficient for this purpose.



Results and conclusion

Figure 7 shows the Rényi Spectra of several fractal sets. Most of the curves were estimated using the sandbox method. To get a sense of the accuracy of the method, the two scale Cantor set, the von Koch curve, and the Sierpinski triangle spectra were plotted. The two-scale Cantor set has good correlation with the exact (theoretical curve) at least for positive values of q. The error might be the result of some random noise in the data. The von Koch curve should be a flat line at 1.26 and the Sierpinski triangle curve should be a flat line at 1.58. The latter especially is underestimated, again especially for negative values of q, but at least they are ordered correctly relative to one another. Thus, this method is producing approximately qualitatively accurate results.

With the accuracy of the estimation method assessed, we can consider the spectra of the sprites. Sprite 1 (figures 1a/1c) has a fairly nice (S-shaped and strictly decreasing) spectrum indicative of multifractalism. Sprite 2 (figures 1b/1d) is less nice, but also seems to indicate multifractalism. The spectrum of a simulated streamer tree (figure 3b) is also nice and indicates multifractal. It is encouraging that the Rényi spectrum of the simulated streamer tree is in the same range as the real sprites. Perhaps the changes mentioned in the fractal simulation section will elimate the difference

Monofractal and multifractal analysis can be very messy and inexact. However, if applied very carefully multifractal analysis provides much richer information than monofractal analysis, and is thus more appropriate to use when validating fractal models of sprites, which are, as we have shown, multifractal objects.

Citations

Carlson, B. E., et al. "Time Domain Simulations of Preliminary Breakdown Pulses in Natural Lightning." Journal of Geophysical Research: Atmospheres, vol. 120, no. 11, 1 June 2015, pp. 5316–5333., doi:10.1002/2014jd022765.

Chhabra, Ashvin, and Roderick V. Jensen. "Direct determination of the f (α) singularity spectrum." Physical Review Letters 62.12 (1989): 1327.

De Bartolo, Samuele G., Roberto Gaudio, and Salvatore Gabriele. "Multifractal analysis of river networks: Sandbox approach." Water resources research 40.2 (2004).

Halsey, Thomas C., et al. "Fractal measures and their singularities: The characterization of strange sets." Physical review A 33.2 (1986): 1141.

Kernighan, Brian W., and Dennis M. Ritchie. The C Programming Language. Prentice Hall, 2016.

Kinsner, Witold. "A unified approach to fractal dimensions." International Journal of Cognitive Informatics and Natural Intelligence (IJCINI) 1.4 (2007): 26-46.

Liu, Ningyu. "Effects of Photoionization on Propagation and Branching of Positive and Negative Streamers in Sprites." Journal of Geophysical Research, vol. 109, no. A4, 2004, doi:10.1029/2003ja010064.

Lopes, Renaud, and Nacim Betrouni. "Fractal and multifractal analysis: a review." Medical image analysis 13.4 (2009): 634-649.

Luque, Alejandro, and Ute Ebert. "Growing Discharge Trees with Self-Consistent Charge Transport: the Collective Dynamics of Streamers." New Journal of Physics, vol. 16, no. 1, 2014, p. 013039., doi 10.1088/1367-2630/16/1/013039.

Niemeyer, Lucian, Luciano Pietronero, and Hans J. Wiesmann. "Fractal dimension of dielectric breakdown." Physical Review Letters 52.12 (1984): 1033.

Pasko, Victor P., et al. "Fractal Structure of Sprites." Geophysical Research Letters, vol. 27, no. 4, 2000, pp. 497–500., doi: 10.1029/1999gl010749.

Pasko, Victor P., and Jeremy J. George. "Three-dimensional modeling of blue jets and blue starters." Journal of Geophysical Research: Space Physics 107.A12 (2002): SIA-12.

Riousset, Jeremy A., et al. "Three-Dimensional Fractal Modeling of Intracloud Lightning Discharge in a New Mexico Thunderstorm and Comparison with Lightning Mapping Observations." Journal of Geophysical Research, vol. 112, no. D15, 2007, doi:10.1029/2006jd007621.

Salem, Mohammed A. "Numerical Solution of Poisson's Equation in Cylindrical Coordinates Using Finite Difference Method." Florida Institute of Technology, Melbourne, Fl, 26 Sept. 2013.

Sanudo, J., et al. "Fractal dimension of lightning discharge." Nonlinear Processes in Geophysics 2.2 (1995): 101-106.

Tél, Tamás, Ágnes Fülöp, and Tamás Vicsek. "Determination of fractal dimensions for geometrical multifractals." Physica A: Statistical Mechanics and its Applications 159.2 (1989): 155-166.

Theiler, James. "Estimating fractal dimension." JOSA A 7.6 (1990): 1055-1073.