# Numerical Error Propagation for the MUSIC Algorithm

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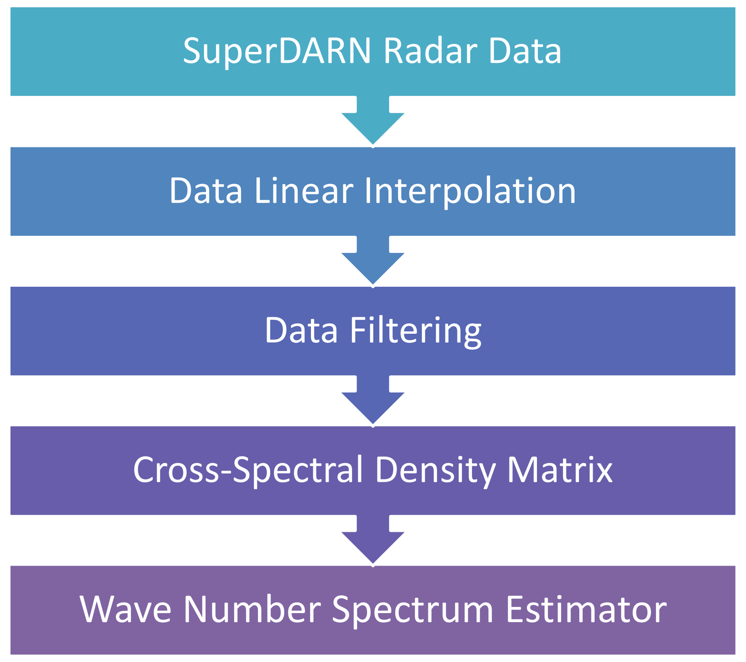
## Introduction

Multiple signal classification (MUSIC) describes theoretical and experimental techniques for determining the parameters of various wavefronts arriving at an antenna array from measurements made on the signals received at the array elements [1]. This technique can be implemented as an algorithm to provide asymptotically unbiased estimates of the strength and direction of the received signals, the polarization, noise/interference, direction of arrival, and several other parameters. It was first used with SuperDARN data by *Samson et al.* [2] and then by Bristow et al. [3]. Further advancements were made to reimplement the algorithm in Python to study the climatology of medium-scale traveling ionospheric disturbances (MSTIDs) by *Frissell et al.* [4]. However, error estimates should be incorporated into the analysis of the MSTID parameters determined by the MUSIC algorithm. This project will revisit this algorithm and explore techniques for integrating numerical error into these measurements.

## The Music Algorithm




- A generalized cross-spectral technique is used for computing the wavelengths and directions of propagation by applying an adaptive beam-forming algorithm.
- Each step of the MUSIC algorithm introduces numerical error.
- We will focus on the statistical error introduced by the wave number spectrum estimator.

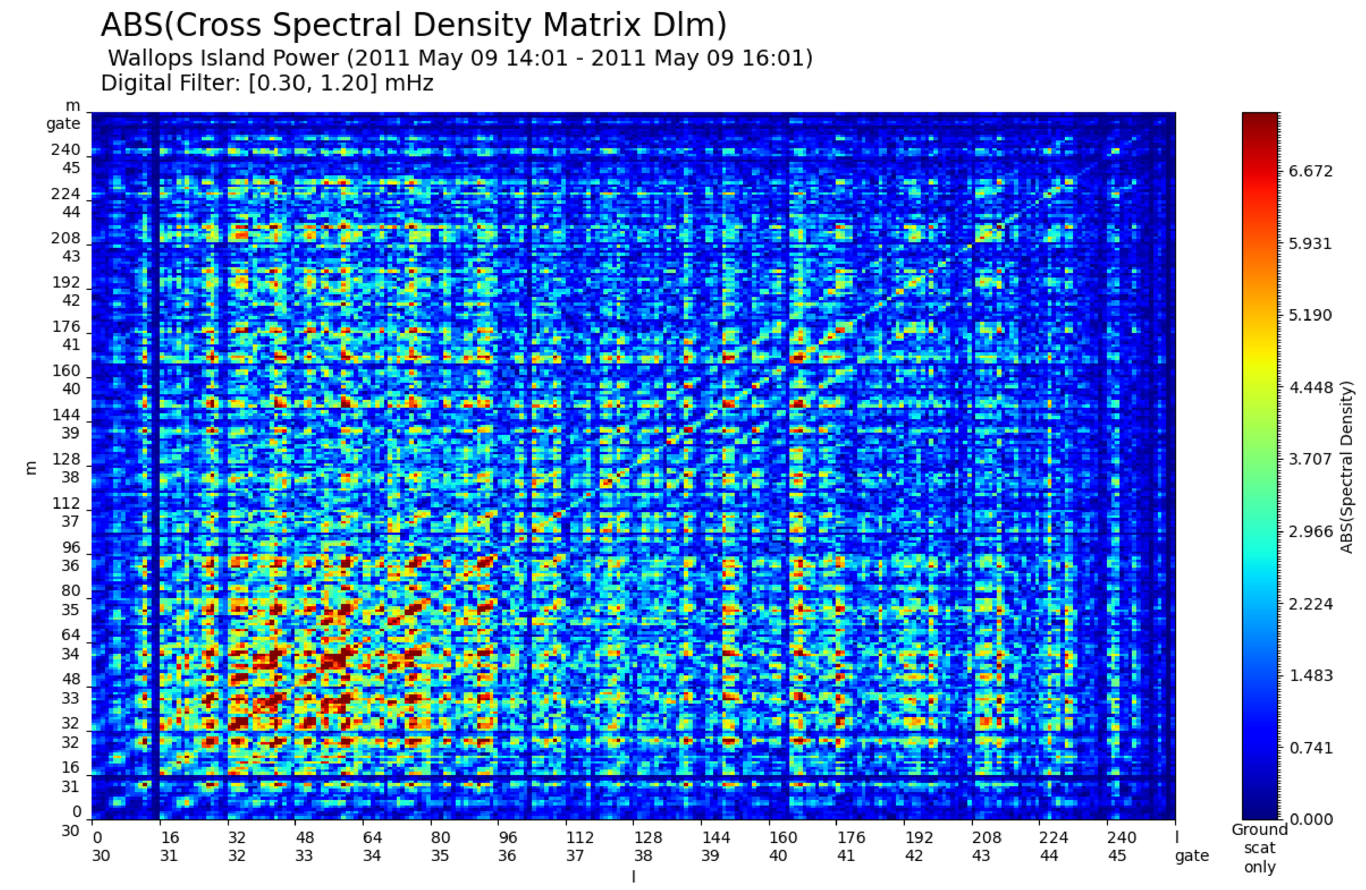
## Cross-Spectral Matrix

- A cross-spectral matrix is a covariance matrix giving the dispersion (variance) between pairs of elements of a given vector.
- The main diagonal of this matrix contains variance directional propagations of the waves. Uncertainties (standard deviations) are the square root of variances in the cross-spectral matrix.

$$D_{lm} = \sum_{\omega} s(\mathbf{r}_l, \omega) s^*(\mathbf{r}_{m,}\omega)$$

- The components of the density matrix $s(\boldsymbol{r}_l, \omega)$ are the Fourier transform in the frequency domain of the signal (time series) at position $\boldsymbol{r}_l$.

The plotted figure of this matrix can be seen below:




## Wave Number Spectrum Estimator

- The wave number spectrum estimator is obtained from the eigenvalues and eigenvectors of the cross-spectral density matrix.

$$P(\mathbf{u}) = \left[ \sum_{j=p+1}^{M} \mathbf{u}^t \mathbf{v}_j \mathbf{v}_j^t \mathbf{u} \right]^{-1}$$

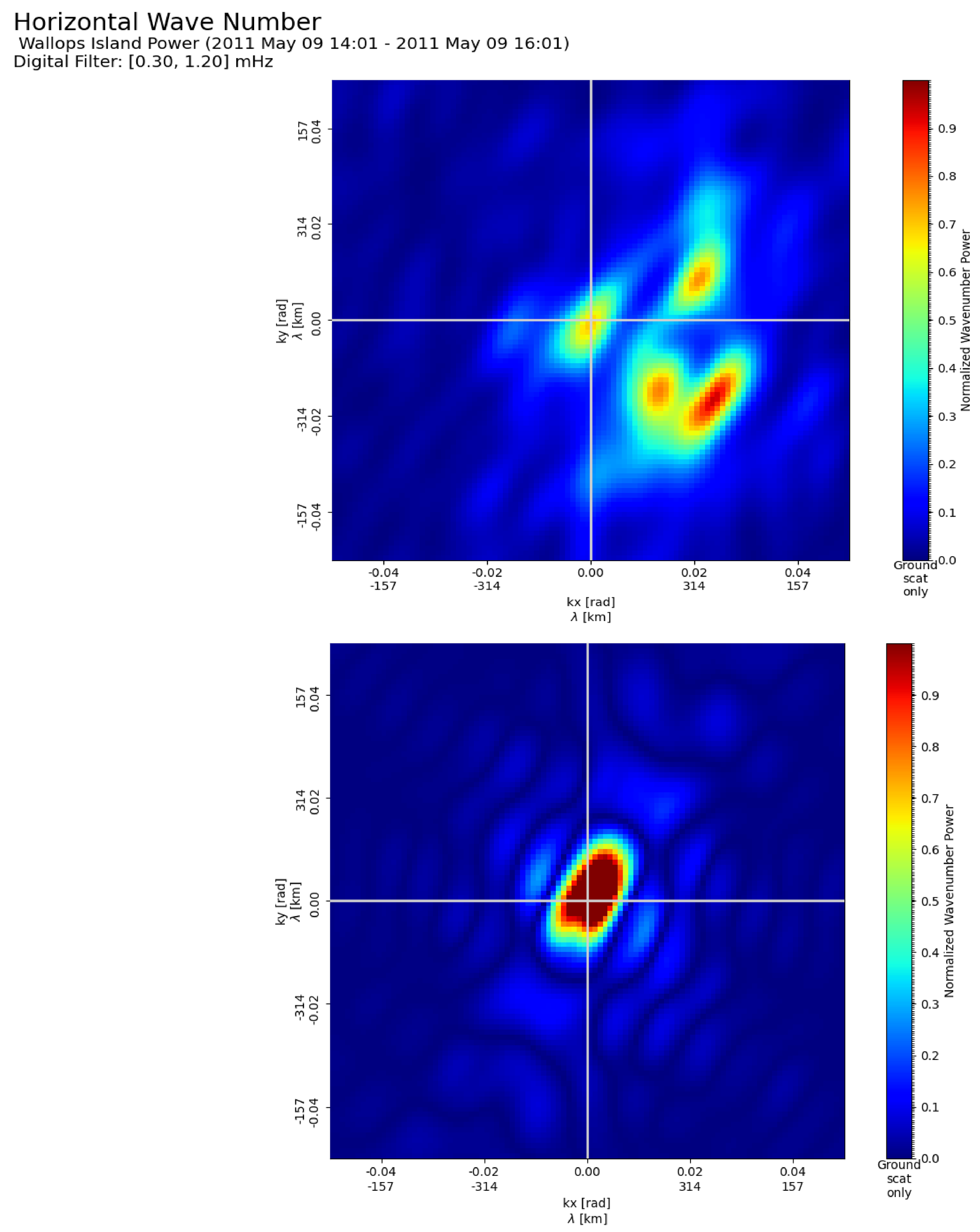
- The vectors $\mathbf{v}_j$ are the eigenvectors corresponding to the $M$ - $p$ smallest eigenvalues of **D**, and the vectors **u** have components $u_m$ representing plane waves

$$u_m = \exp\left[ \mathrm{i}\left( k_x x_m + k_y y_m - \omega t \right) \right]$$

## Error Propagation Analysis

- We will account for the mathematical error in the MUSIC algorithm by looking at the wave number spectrum estimator equation.
- Our first approach assumes that the **u** vectors have no errors as they are computed by dividing the wave number plane into an equispaced grid.
- We also assume that the radar ranges (that the radar ranges $x_m$ and $y_m$) are obtained error-free.
- The eigenvectors $\mathbf{v}_j$ of matrix **D** would have errors corresponding to the matrix.
- Error propagation is then performed over the different matrix operations in the equation.
- Finally, a new plot is created showing the most significant error areas in the wave number spectrum plot.

## Wave Number Spectrum and Error Plots




## Conclusions / Future Work

- Our method showed that features with small wave numbers may have significant error, so they are not reliable or identifiable.
- The error obtained by this technique is statistical, showing the amount by which, an observation differs from its expected value.
- More troubleshooting of the algorithm we developed to calculate the error must occur. We are in the early stages of this study.
- We need to develop new error analysis strategies that propagate the error introduced by the MUSIC algorithm's different steps.

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### References

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