Exploring Data Using Fourier Techniques

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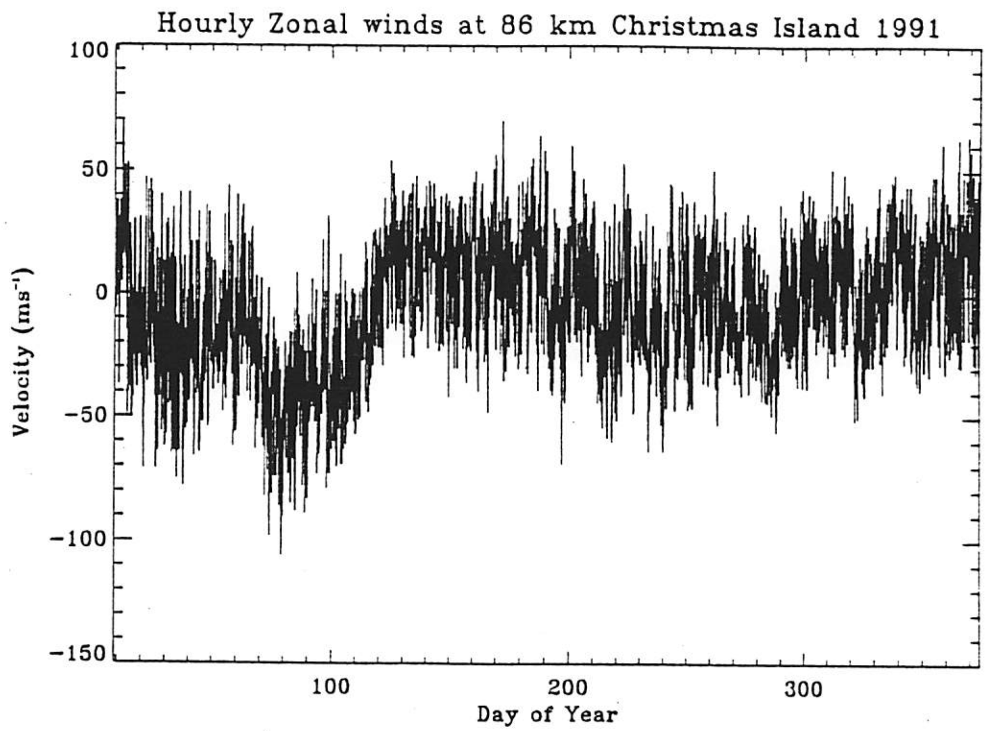
This part of the course shows how conventional Fourier techniques can be used to explore data. The methods are generally applicable to either temporal or spatial data sets.

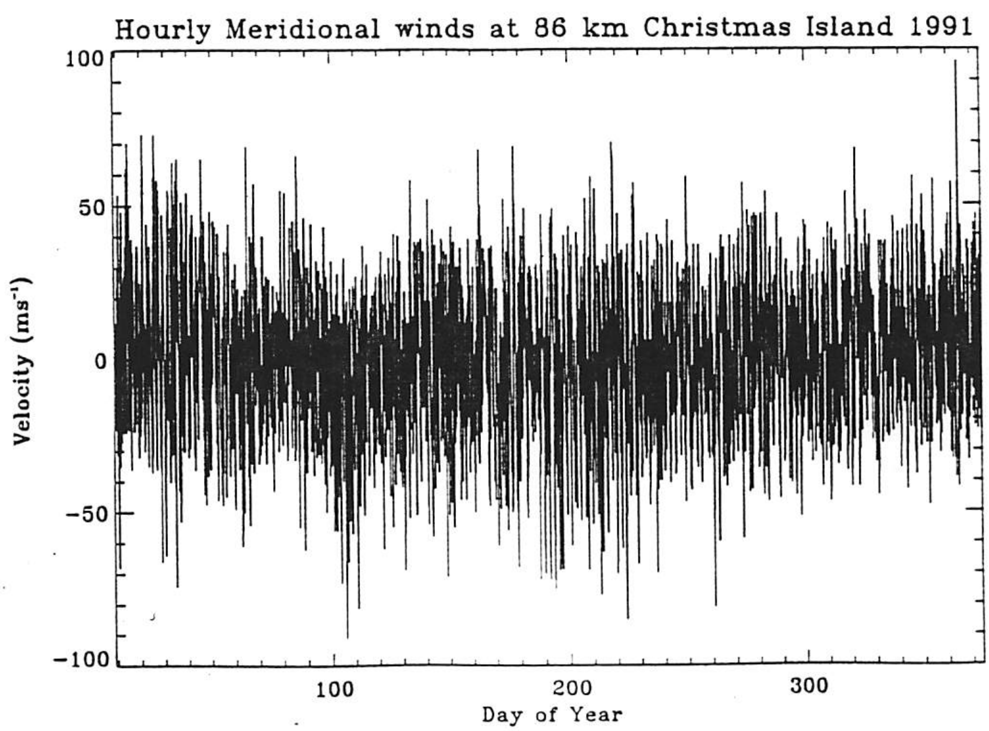
- **Moving power spectra.**

- **Filters.**

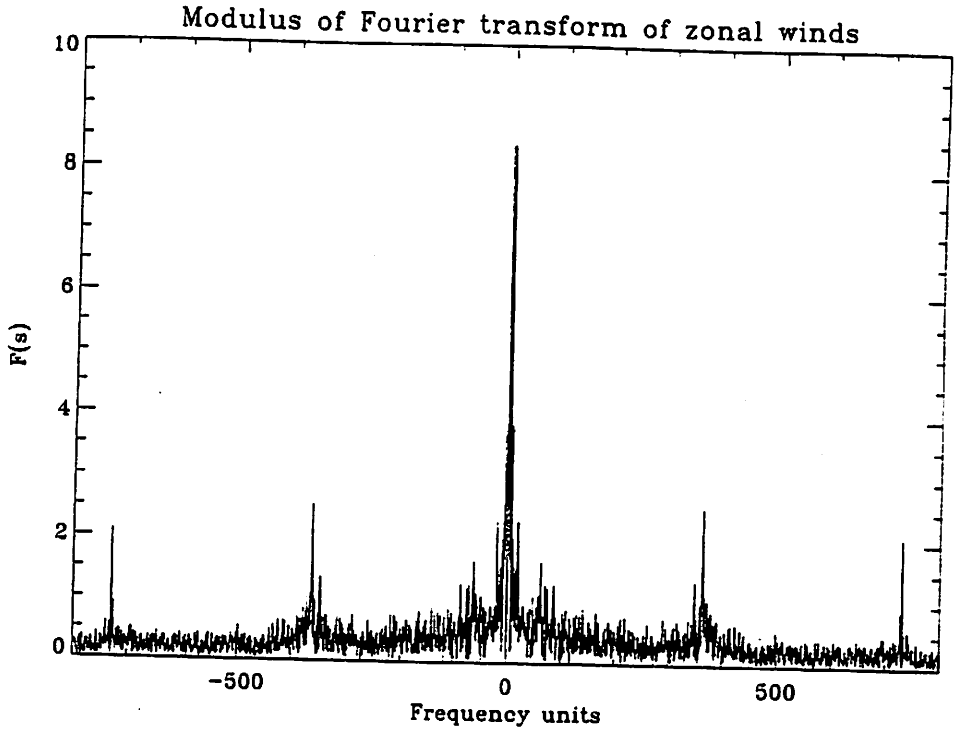
- **Complex demodulation.**

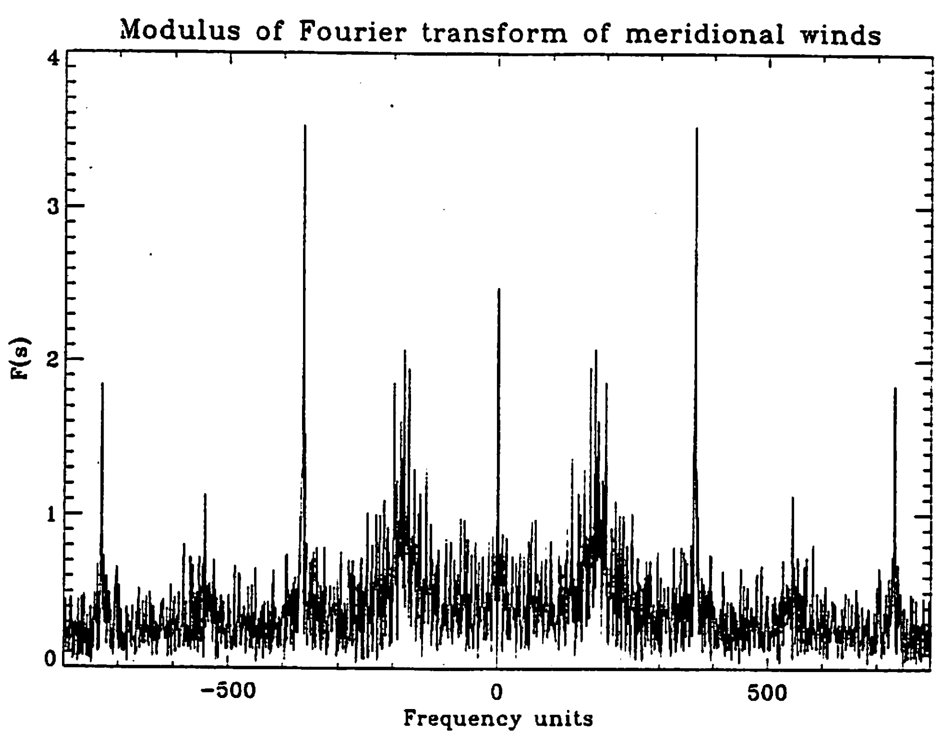
Examples are drawn from a one-year (8760 x 1 h points) time series of zonal and meridional wind components.

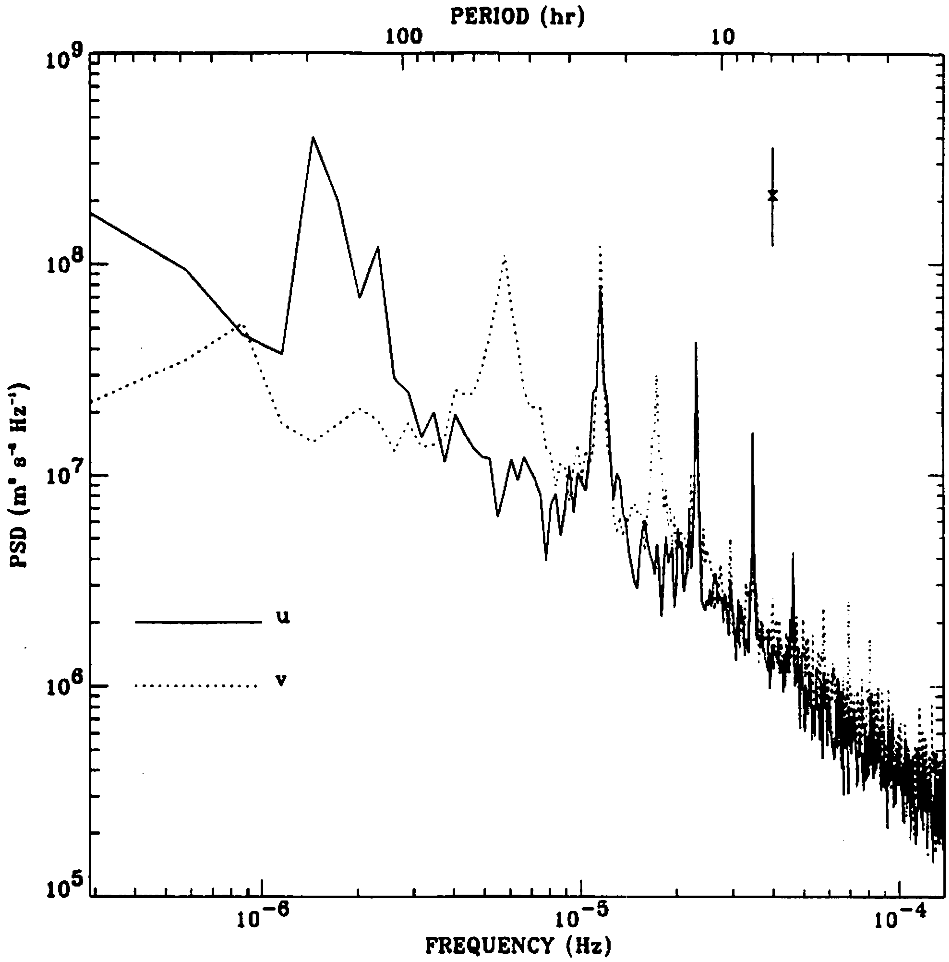

Hourly Zonal winds at 86 km Christmas Island 1991
100
50
0
-50
-100
-150
Velocity (ms⁻¹)
100
200
300
Day of Year


Hourly Meridional winds at 86 km Christmas Island 1991
100
50
0
-50
-100
Velocity (ms⁻¹)
100
200
300
Day of Year

Raw Spectra: $\Delta f = 1/8760$ h-1



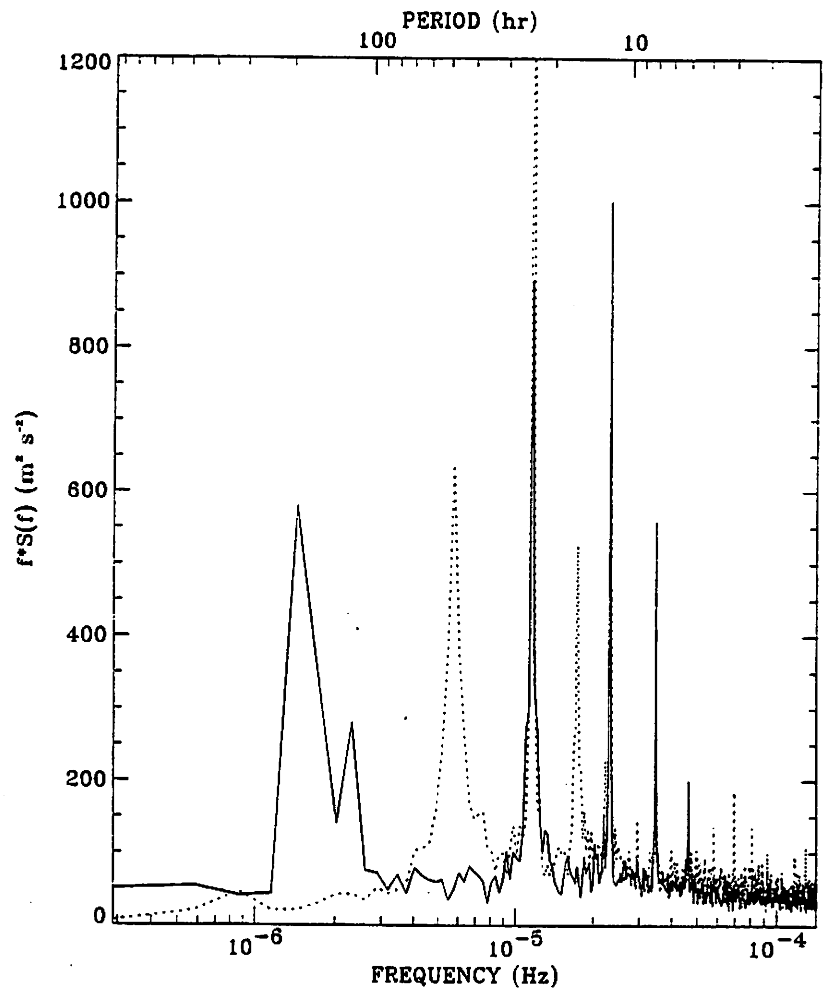




<u>Power Spectra</u>

(Ref.: *Numerical Recipes, ch. 13; Jenkins and Watts*)

- **Welch window used.**
- **Block averaged to reduce variance.**
- **Nine 40-day blocks used, overlapped by 50% to give 17 overlapping segments (c.f. *NR* §13.4).**
- **~26 degrees of freedom give 95% confidence limits (*JW*) of 0.61 and 1.8**
- **On a log plot the 'error bars' are uniform.**



Another way to display a power spectrum is in "Energy-preserving" form, where *f S(f)* is plotted against log(*f*).

- **Area under given segment of width Δ(log*f*) is proportional to variance.**

$$f.S(f)\Delta(\log f) = f.S(f)\frac{\Delta f}{f} = \sigma_\Delta^2$$

Moving Power Spectra
(Sliding FFTs)

1. Select data with window of length T.

2. Compute spectrum, $S(f)$.

3. Assign $S(f)$ to center of window.

4. Step window on by amount ΔT.

5. Repeat 2. for data interval $\Delta T \rightarrow T + \Delta T$, etc.

6. Finish with 2-d array of $S(f)$ vs t.

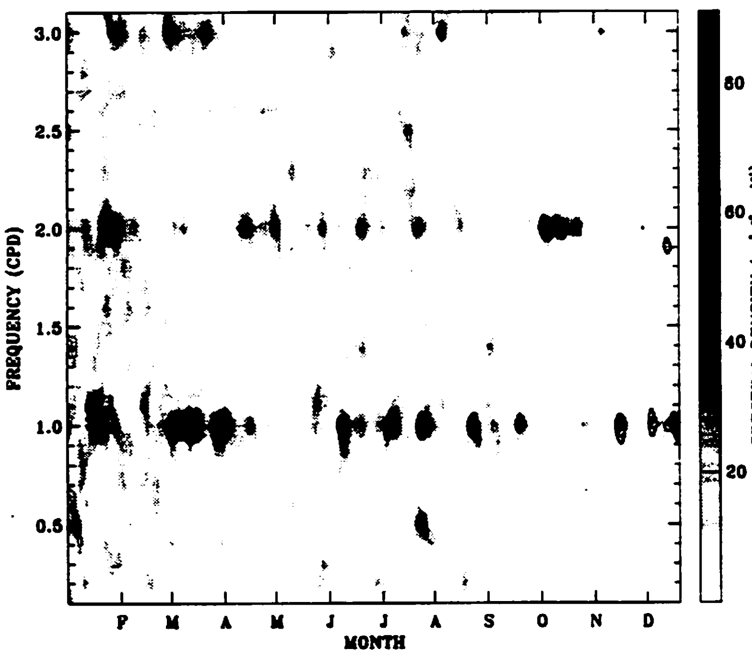
Time Resolution vs Frequency Resolution

Some Examples

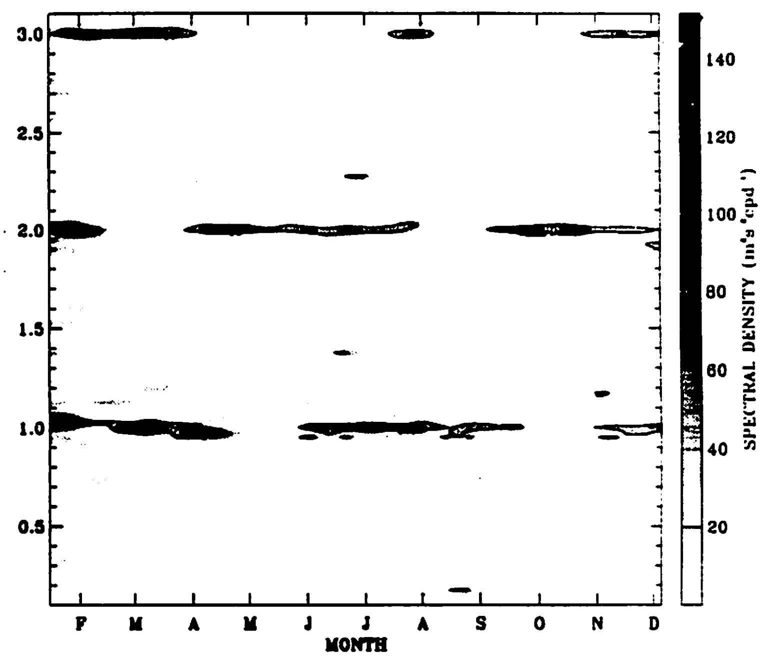
ΔT = 10 day

ΔT = 30 day

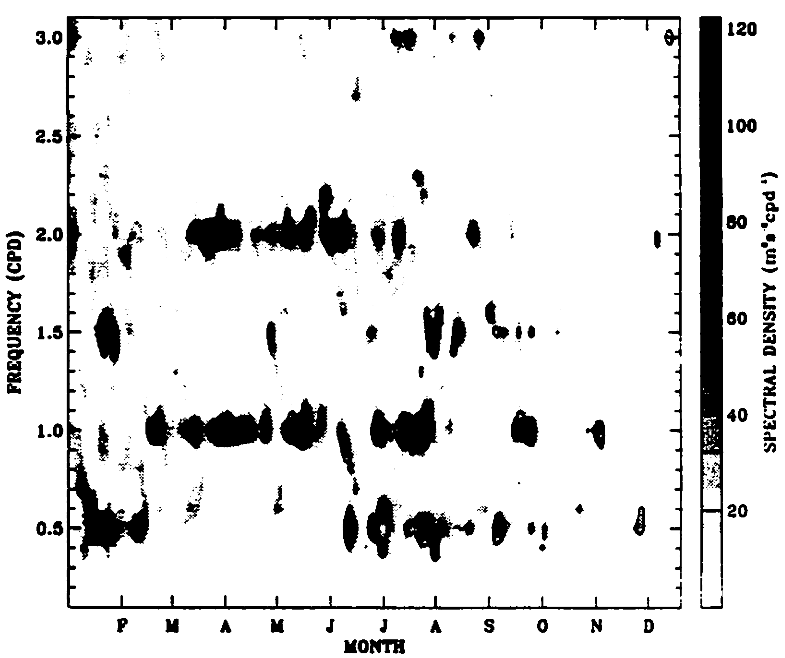
POWER SPECTRUM ZONAL WINDS AT 86 KM



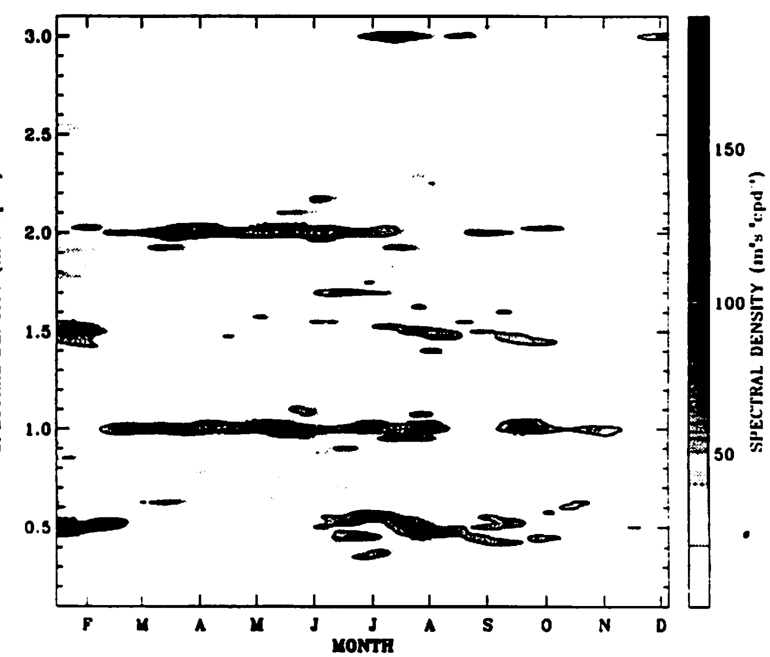
POWER SPECTRUM ZONAL WINDS AT 86 KM



POWER SPECTRUM MERIDIONAL WINDS AT 86 KM



POWER SPECTRUM MERIDIONAL WINDS AT 86 KM

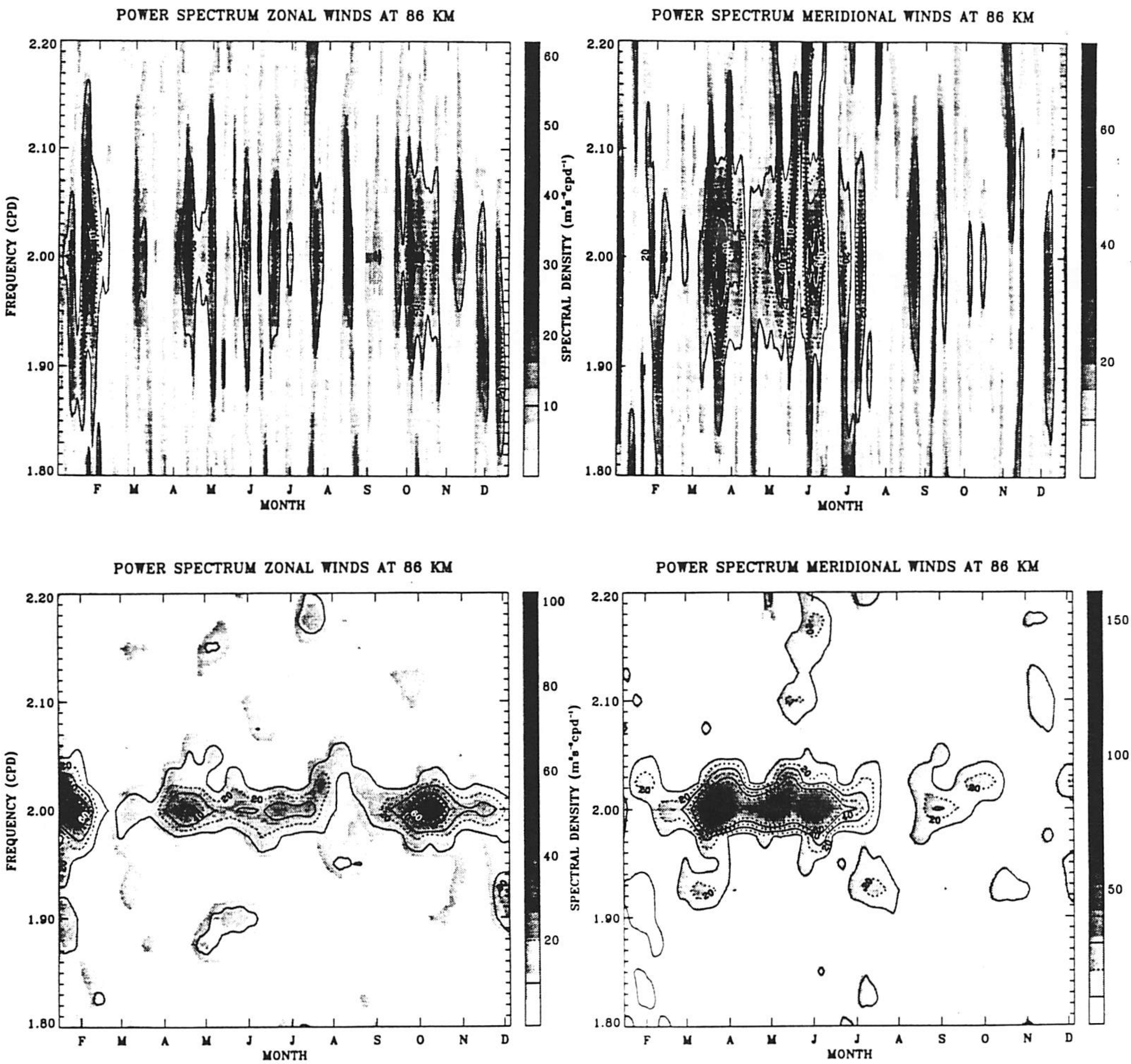


- **Resolution depends on window.**

  $\Delta T = 10$ day $\Rightarrow \Delta f = \sim \pm 0.15$ cpd (Welch window)

  $\Delta T = 40$ day $\Rightarrow \Delta f = \sim \pm 0.04$ cpd (Welch window)

- **Watch out for side-lobes.**



Filters

Time-Domain Filters

A simple filter is *n*-point weighted moving average:

$$g_i(t) = \frac{1}{W} \sum_{j=0}^{n-1} w_j f_{i+j-n/2} \text{ where } W = \sum_j w_j$$

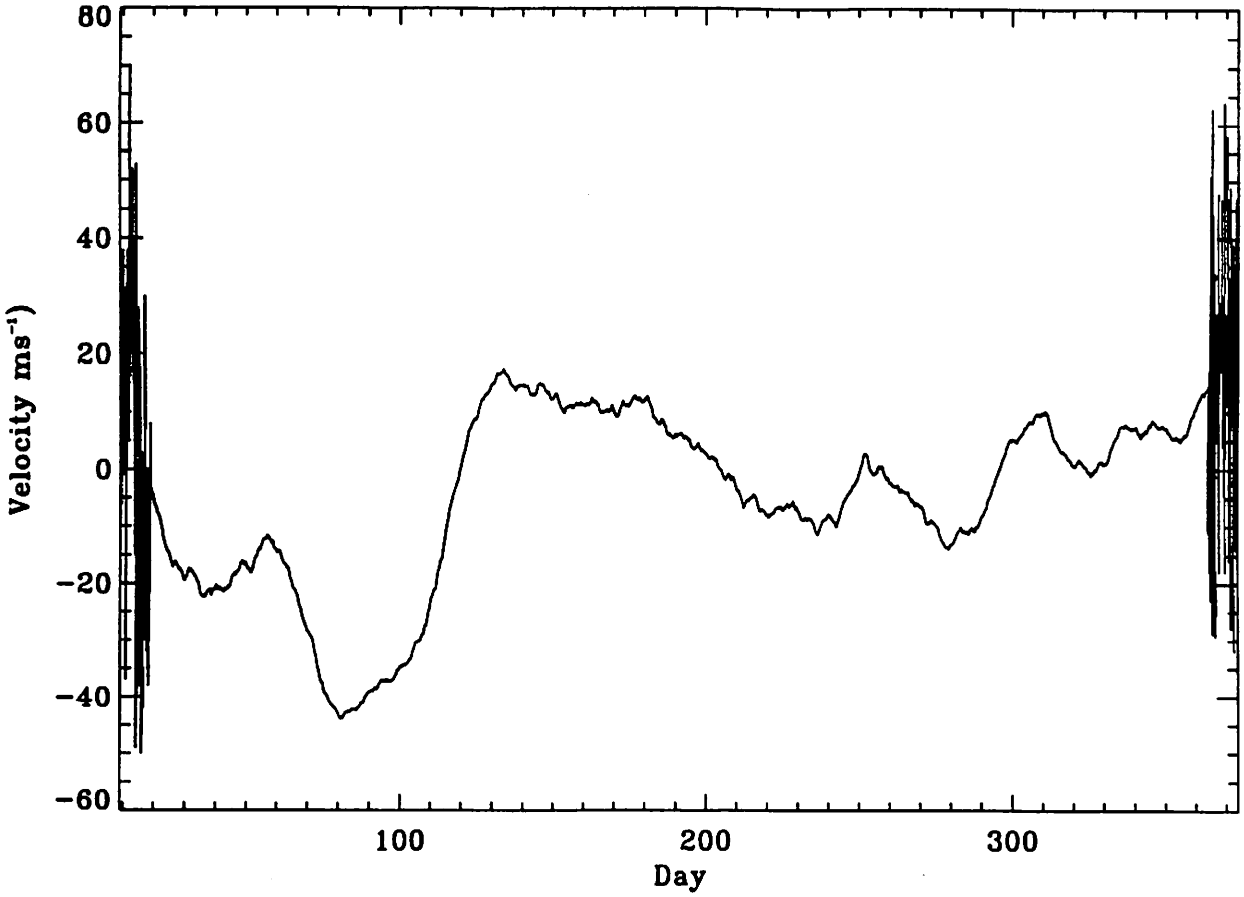
for $i = n/2.....N-n/2$

w_j are weights.

(e.g 'Triangular-average' is often used to lightly smooth data: $w_j = ¼, ½, ¼$)

Simplest example is Rectangular or Box-car average:

$w_j = 1$ and $W = n$



Zonal winds smoothed with a 481-point (20-day) moving average.

Note loss of 240 points (10 days) of data at each end.

Also note structure with time scales of less than 20 d.

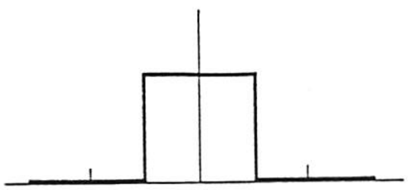
Remember:

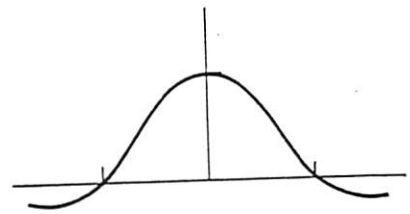
"Convolution (⊗) in one domain is equivalent to multiplication in the other domain"

$$f \otimes g \leftrightarrow F.G$$

or

$$f.g \leftrightarrow F \otimes G$$

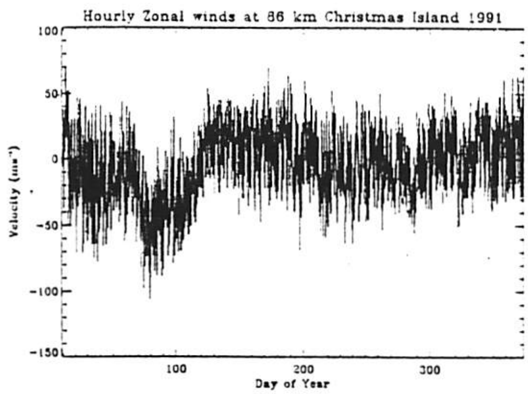


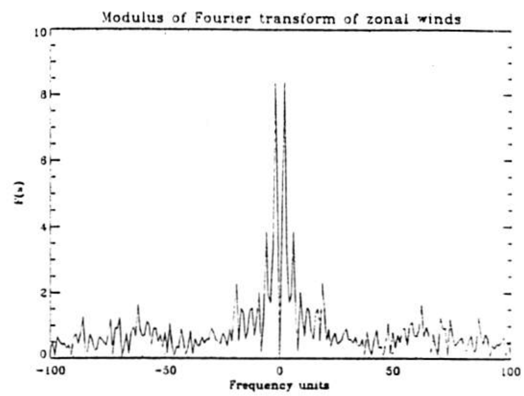


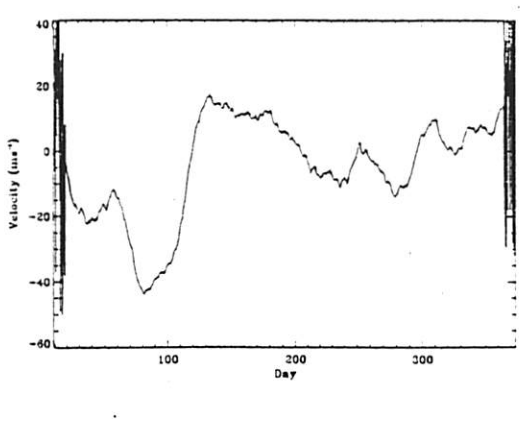
convolve

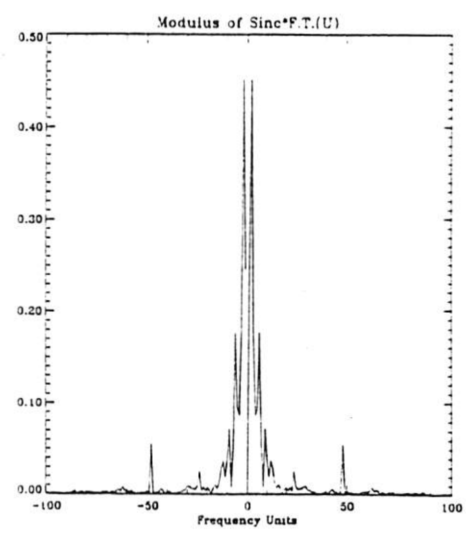
⊗

multiply









The action of a filter on an input signal, y(t), can be summarized as:

Time Domain		**Frequency Domain**
$y(t)$	$\leftrightarrow$	$Y(f)$
$\otimes$		***multiply***
$h(t)$	$\leftrightarrow$	$H(f)$
$\downarrow$		$\downarrow$
$g(t)$	$\leftrightarrow$	$G(t)$

H(f) **is the** ***Transfer Function.*** **Easily found if**

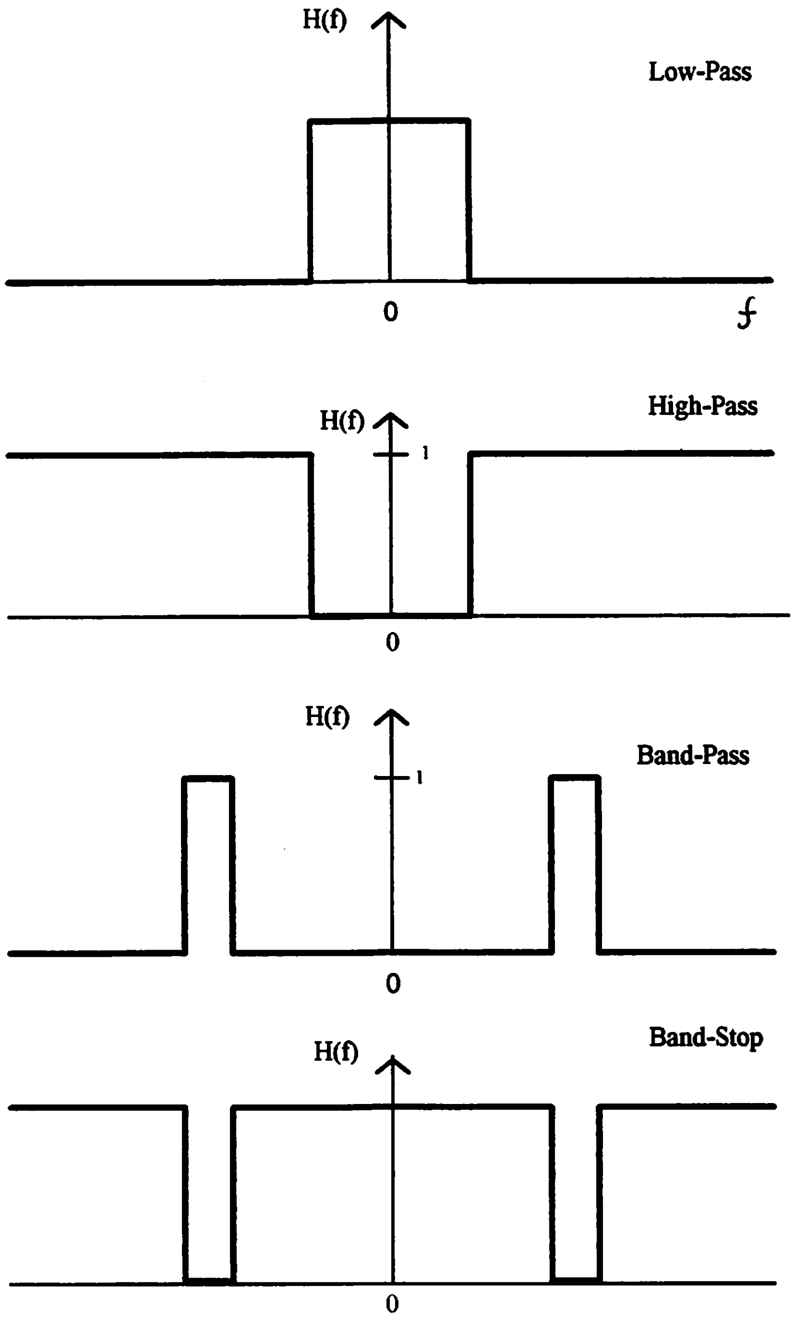
$$y(t) = \delta(t) \leftrightarrow Y(f) = 1$$

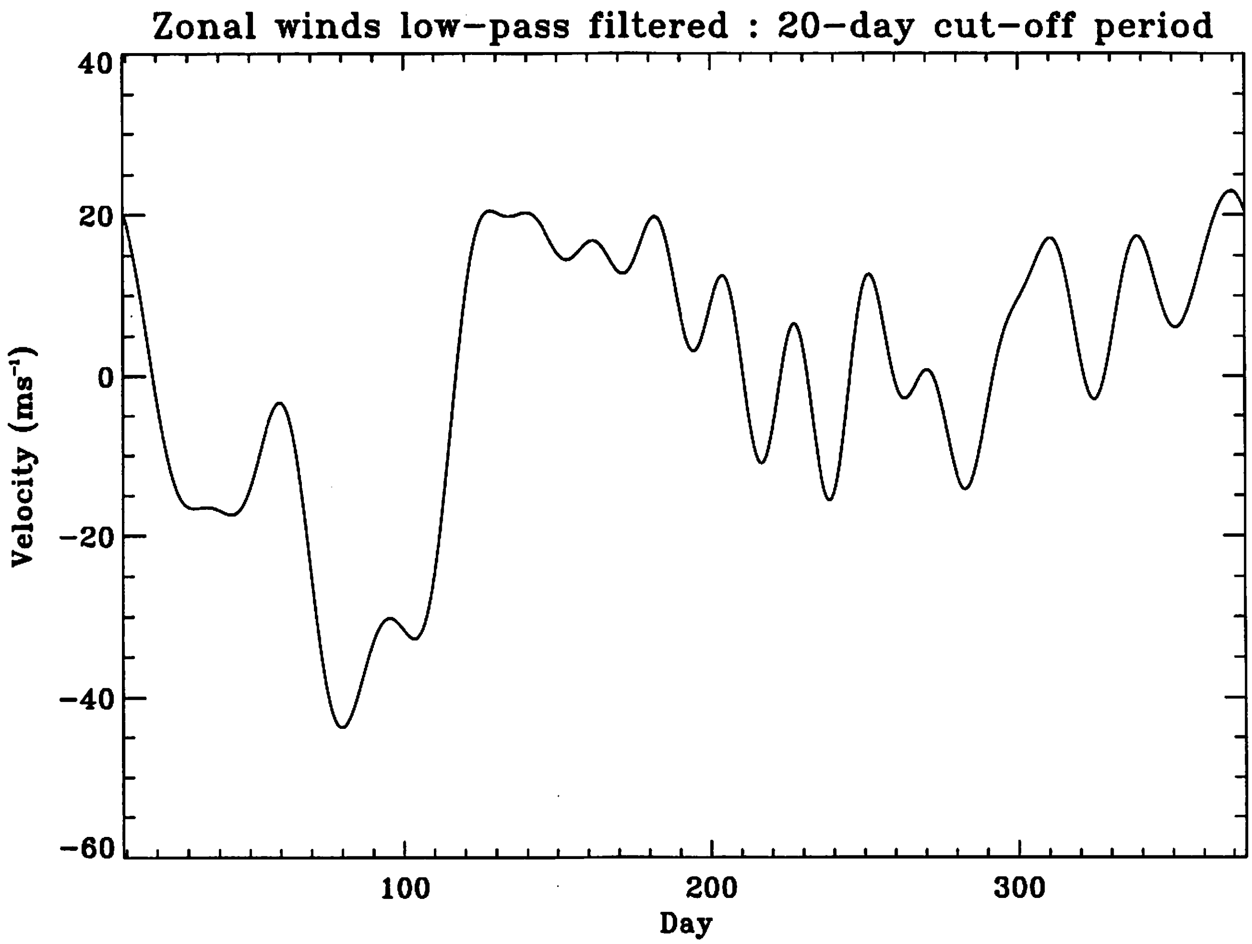
h(t) **is** ***Impulse response.***

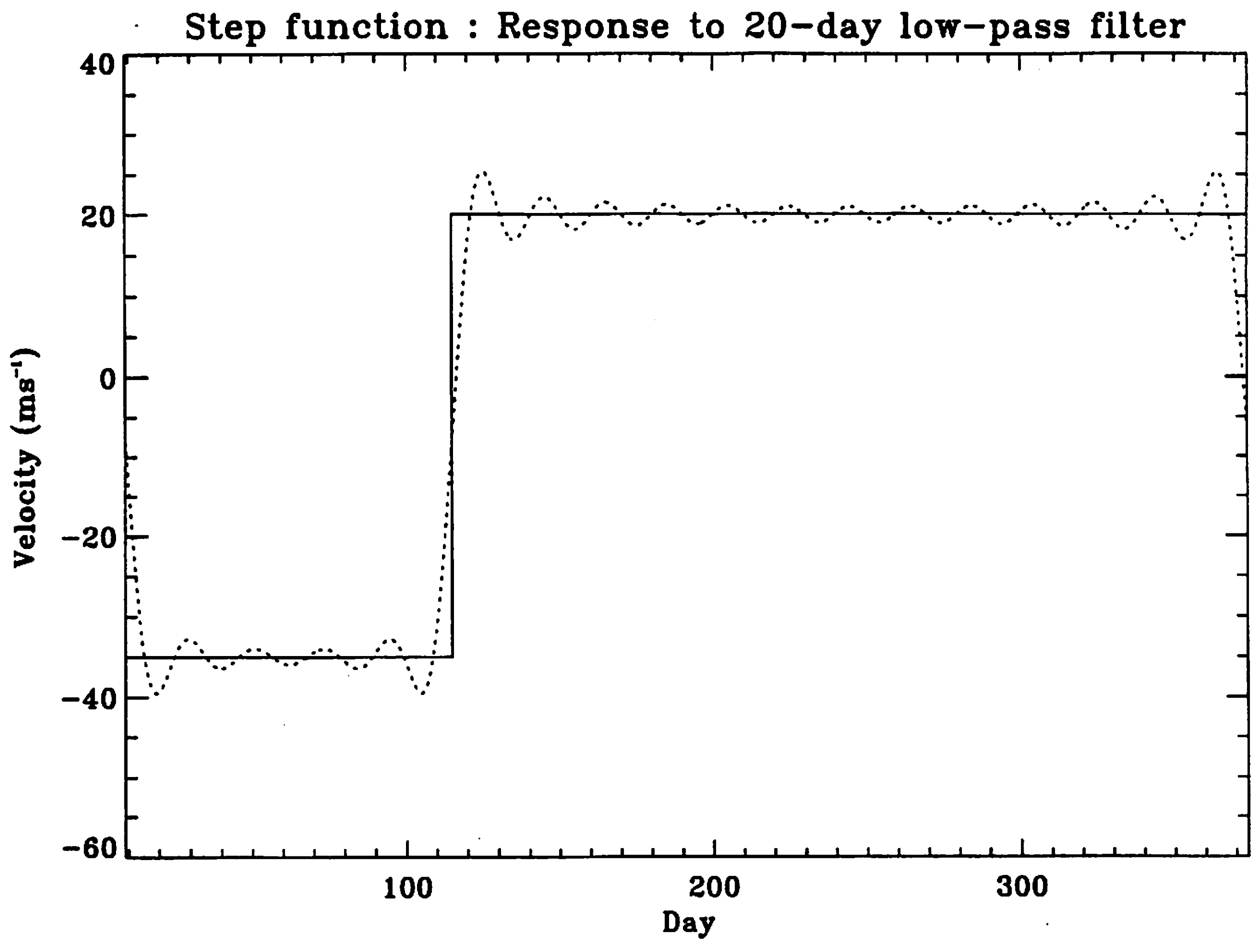
or specify filter by ***Step response, S(f):***

$$H(f) = 2\pi i S(f)$$

Ideal Filters




Zonal winds low−pass filtered : 20−day cut−off period
40
20
0
−20
−40
−60
Velocity (ms⁻¹)
100
200
300
Day

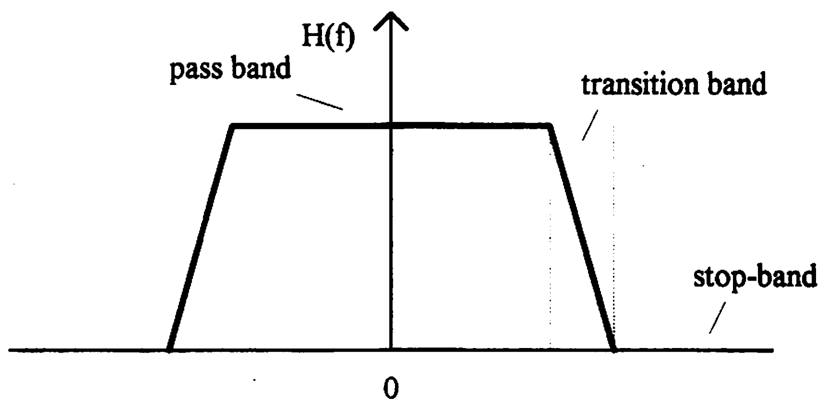


- Note overshoot and ringing - Gibbs phenomenon.

- End effects due to windowing of data.

Tapered or Transition Filters

Tapered Low-Pass



See Forbes, Elgar, Rabiner and Gold, and Kuc, for further discussion.

Simple taper is raised-cosine function (see Kuc, ch 9)

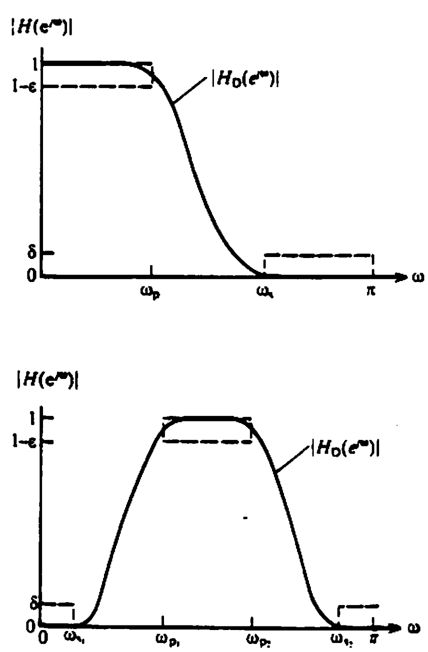
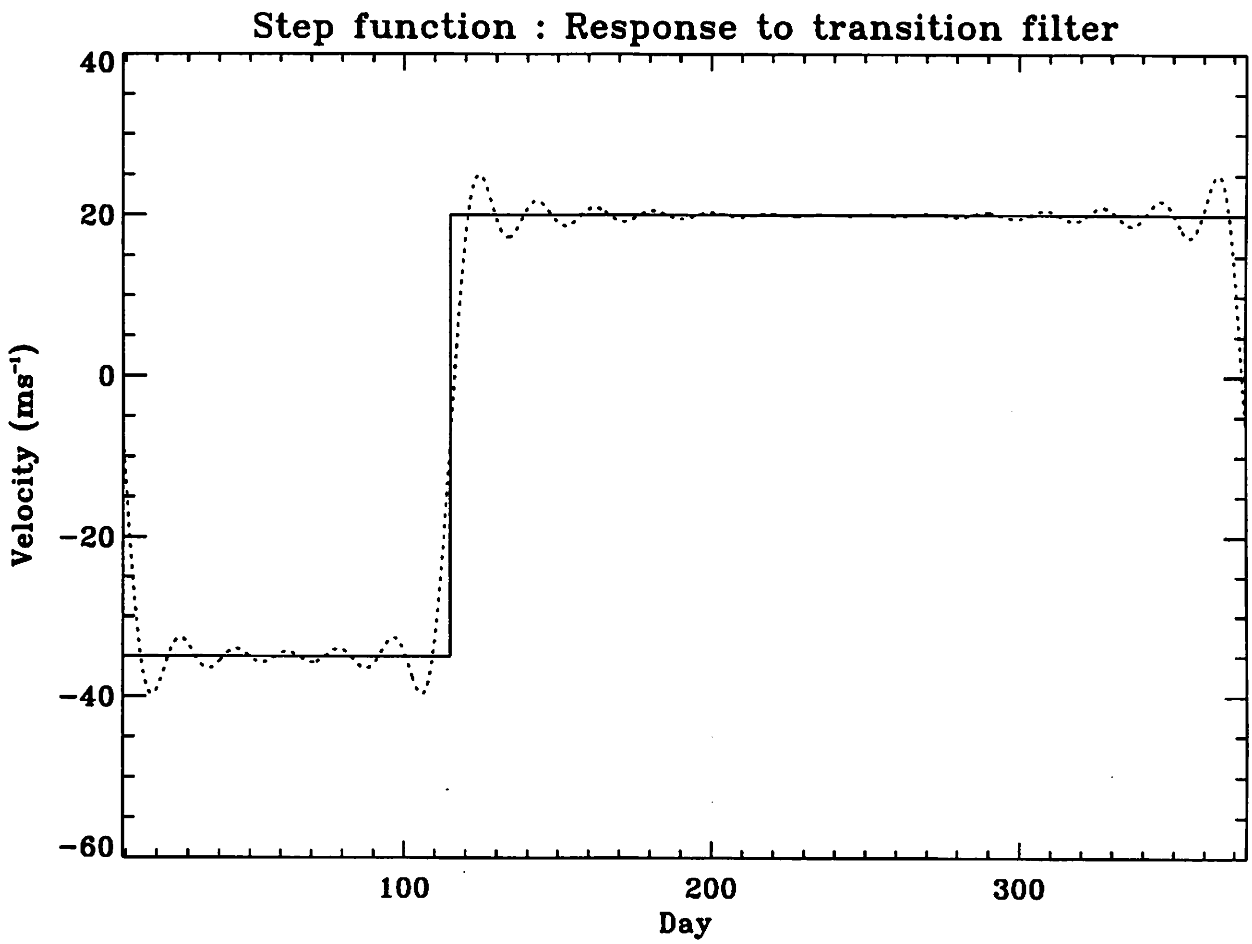
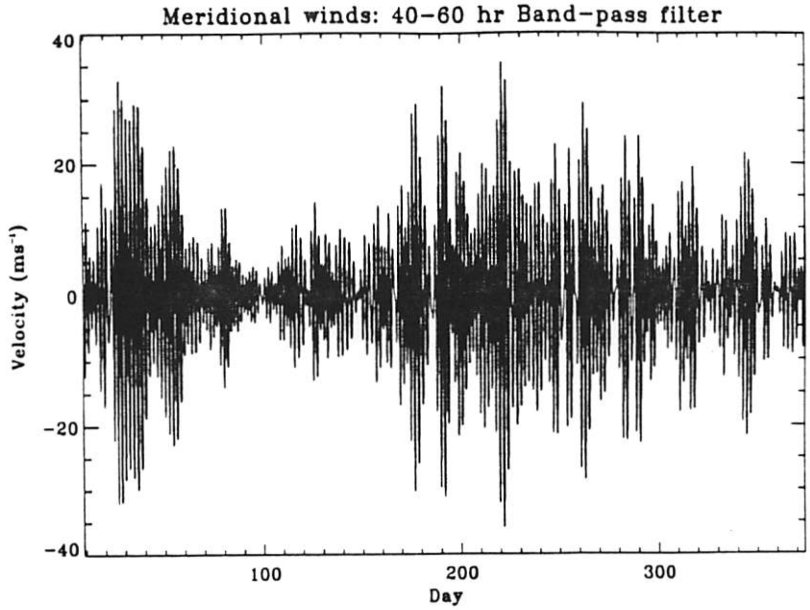
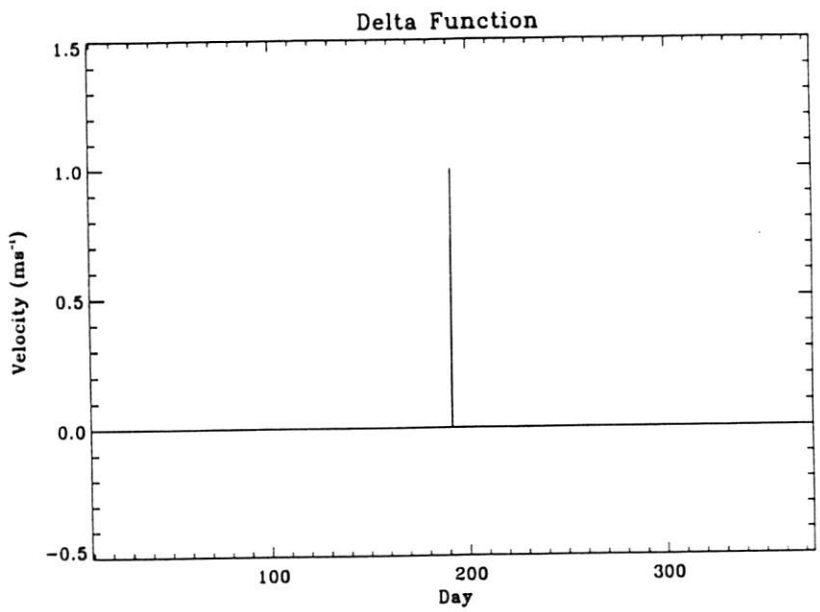


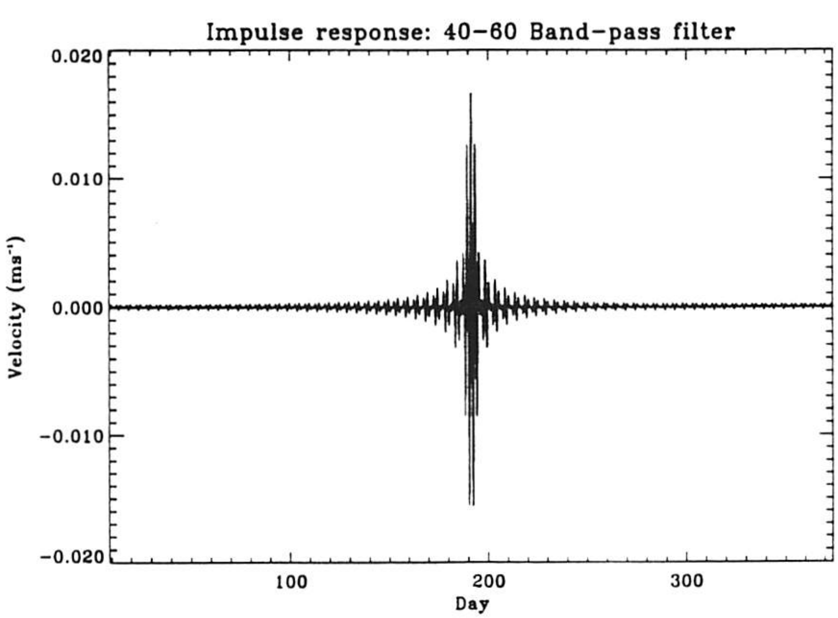
FIGURE 9-33
The desired magnitude response for lowpass and bandpass filters is obtained by connecting the passband and stopband with a raised-cosine function.



Step function : Response to transition filter
Velocity (ms⁻¹)
40
20
0
-20
-40
-60
100
200
300
Day


Meridional winds: 40-60 hr Band-pass filter
Velocity (ms⁻¹)
40
20
0
-20
-40
100
200
300
Day


Delta Function
Velocity (ms⁻¹)
1.5
1.0
0.5
0.0
-0.5
100
200
300
Day


Impulse response: 40-60 Band-pass filter
Velocity (ms⁻¹)
0.020
0.010
0.000
-0.010
-0.020
100
200
300
Day

Filtering in the Frequency Domain

(See also *NR, §13.5*)

- **Define filter for both positive and negative frequencies.**

- **For real data chose $H(f)$ that is real and even in frequency i. e. $H(-f) = H(f)^*$.**

- **Sharp edges in $H(f)$ will produce ringing (Gibb's phenomenon) for impulsive input. Test filter before using. May need to taper edges.**

- **Forward transform time series, $y(t)$, to give $Y(f)$.**

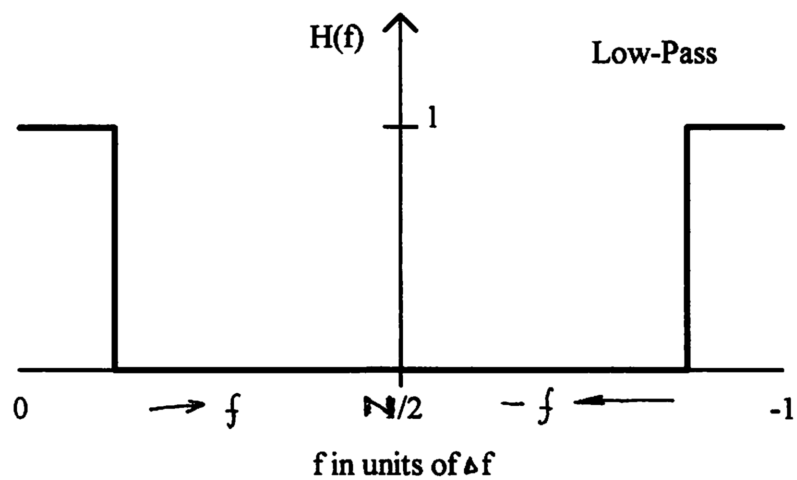
- **Form product of $Y(f)$ and $H(f)$.**

- **Inverse transform $G(f)$ to give $g(t)$.**

- **REMEMBER that FFT orders transform in frequency space as:**

  0, Δf, 2Δf,.........±NΔf/2,.........-2Δf, -Δf

  where Δf = 1/NΔt



COMPLEX DEMODULATION

(Ref: *Bloomfield, ch. 6)*

A convenient way to investigate variations of amplitude and phase (frequency).

Consider the time series:

$$f(t) = A(t)\cos(\omega t + \phi(t)) \text{ or } f(t) = A(t)\cos(\omega t)$$

where A and ϕ (or ω) are slowly varying functions of time.

Complex demodulation allows A and ϕ to be described as a function of t.

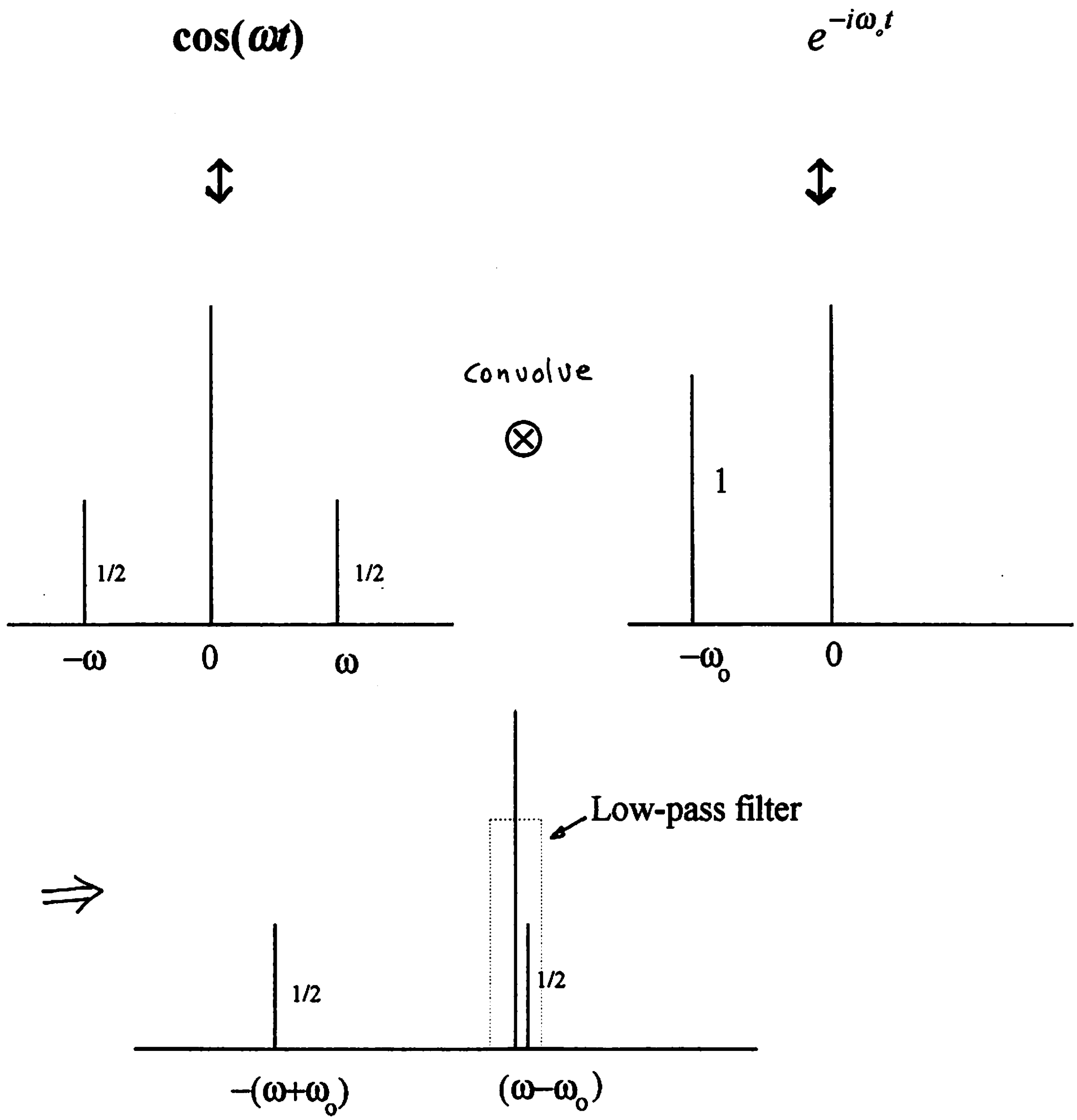
First consider the simple function

$$f(t) = \cos(\omega t)$$

Multiply by $e^{-i\omega_o t}$, where $f_o = \omega_o/2\pi$ is the frequency we wish to demodulate about, to get:

$$g(t) = \cos(\omega t)e^{-i\omega_o t}$$

In the frequency domain this is equivalent to:



After low-pass filtering we get:

$$g'(t) = \tfrac{1}{2}\cos\{(\omega - \omega_o)t\}$$

(note, have to double g' to recover amplitude).

To apply:

1. **Choose center frequency f_o.**

2. **Multiply time series by $e^{-i\omega_o t}$**

3. **Low-pass filter to give g'.**
   Width of filter determines rate at which temporal variations can be studied.

4. **To recover amplitude and phase:**

$$A(t) = 2.|g'(t)|$$

$$\phi(t) = \tan^{-1}\left(\frac{imag(g')}{real(g')}\right)$$

Example: Time series with 8760 points ('hrs').

$f = 1/48\ h^{-1}$

$f_o = 1/50\ h^{-1}$.

Filter bandwidth of the filter is 240 points (i.e. '10 days').

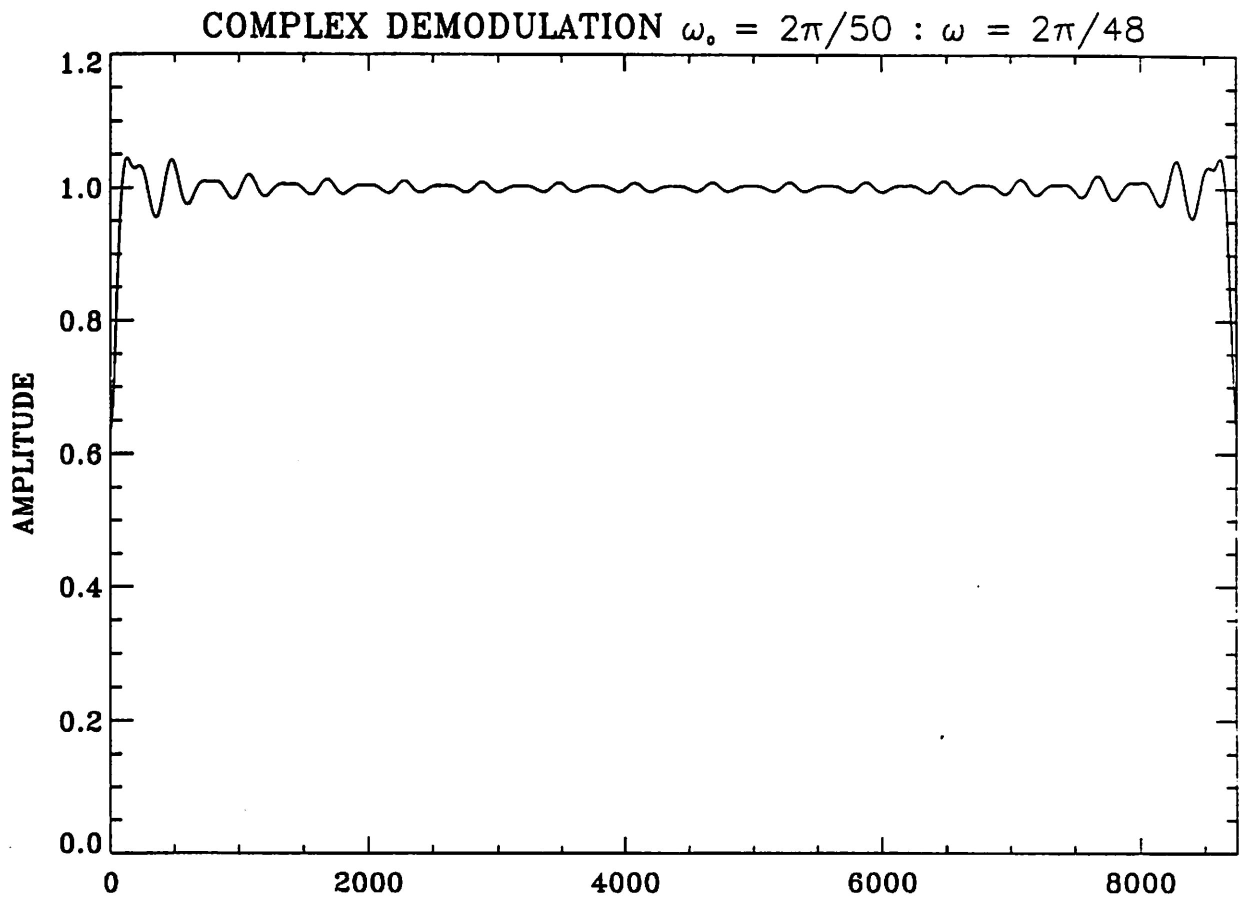
Note:

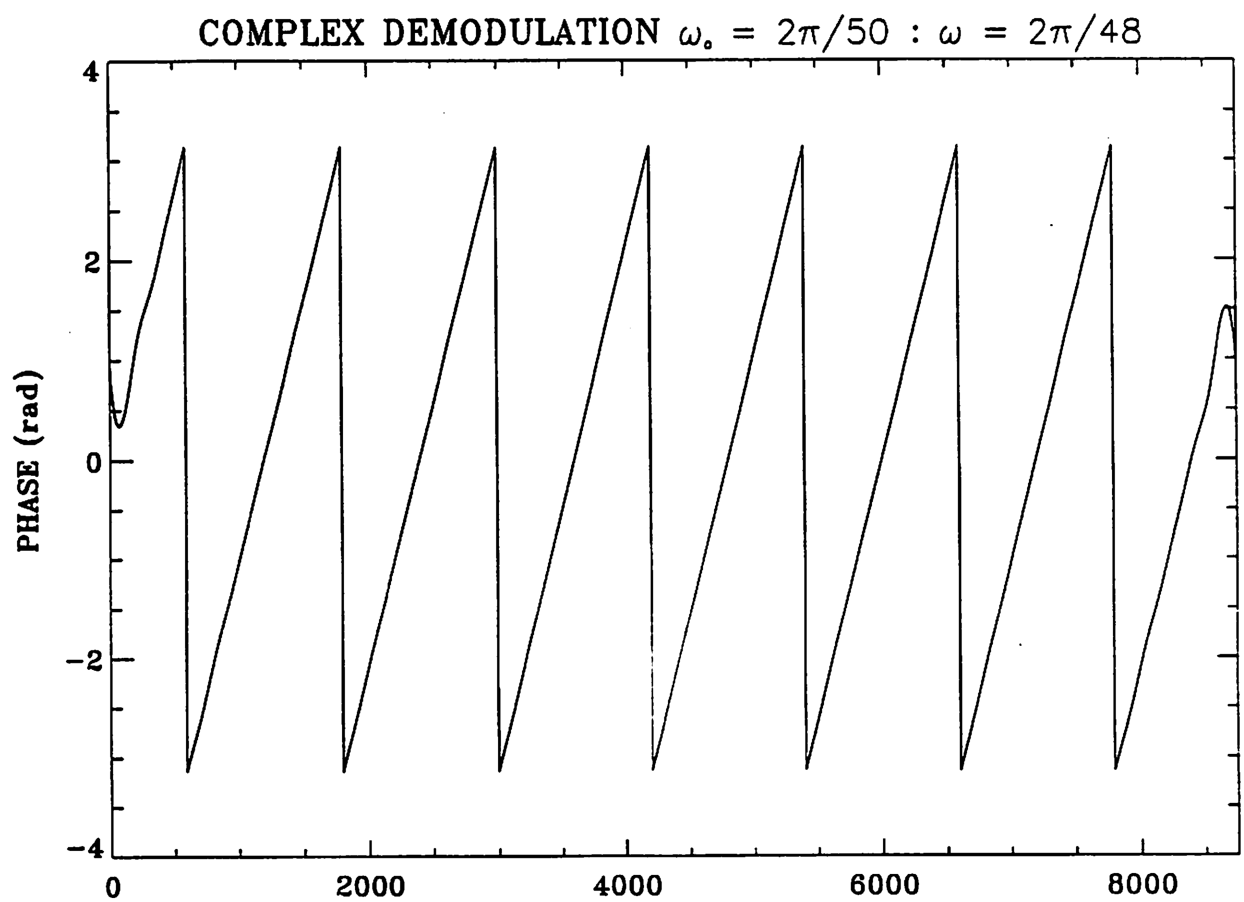
End effects due to filter.

Positive phase shifts with slope of 1 cycle per 1200 points - equivalent to frequency difference of 0.02 h^{-1}.

The phase gradient is a measure of the 'local' frequency difference from demodulation frequency, f_o.

$$f - f_o = \frac{1}{2\pi}\frac{d\phi}{dt}$$

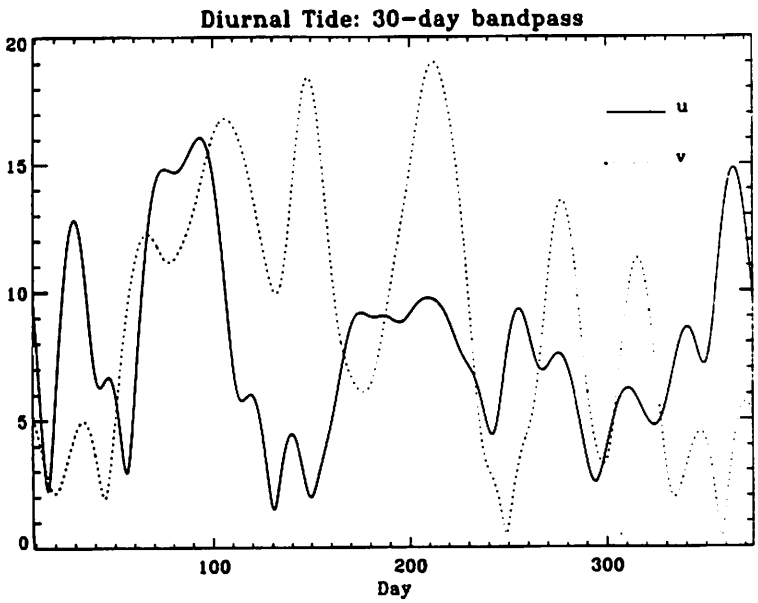

COMPLEX DEMODULATION $\omega_o = 2\pi/50 : \omega = 2\pi/48$
AMPLITUDE
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
2000
4000
6000
8000

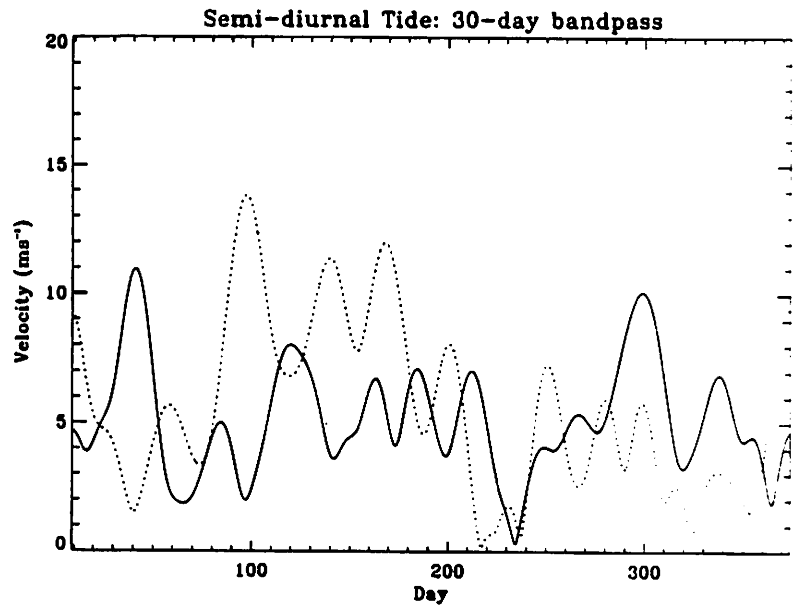

COMPLEX DEMODULATION $\omega_o = 2\pi/50 : \omega = 2\pi/48$
PHASE (rad)
4
2
0
-2
-4
0
2000
4000
6000
8000

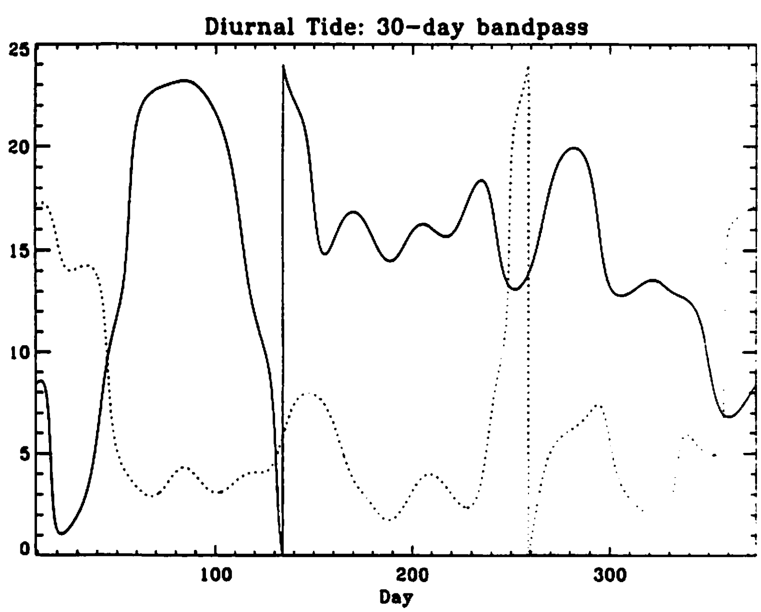
Complex Demodulation can be thought of as a "local" harmonic analysis i.e., a moving fit of

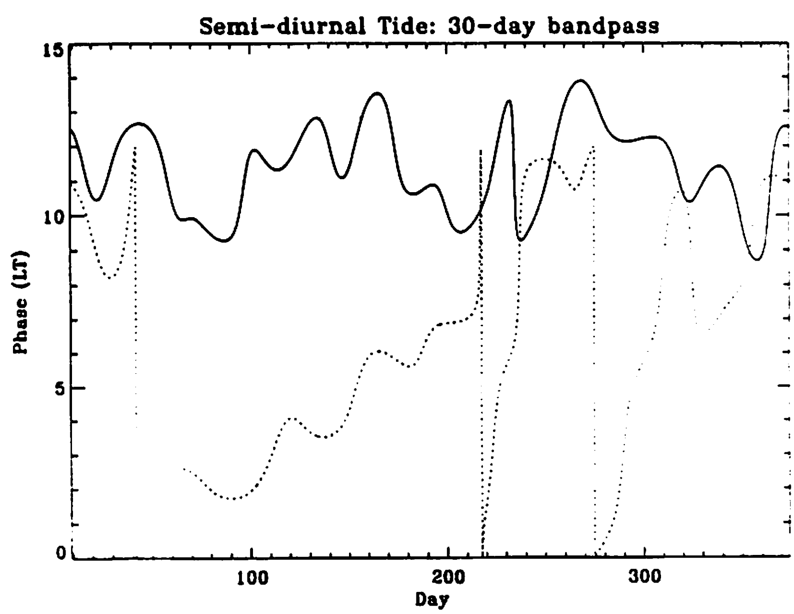
$$f(t) = A(t)\cos\{\omega t + \phi(t)\},$$

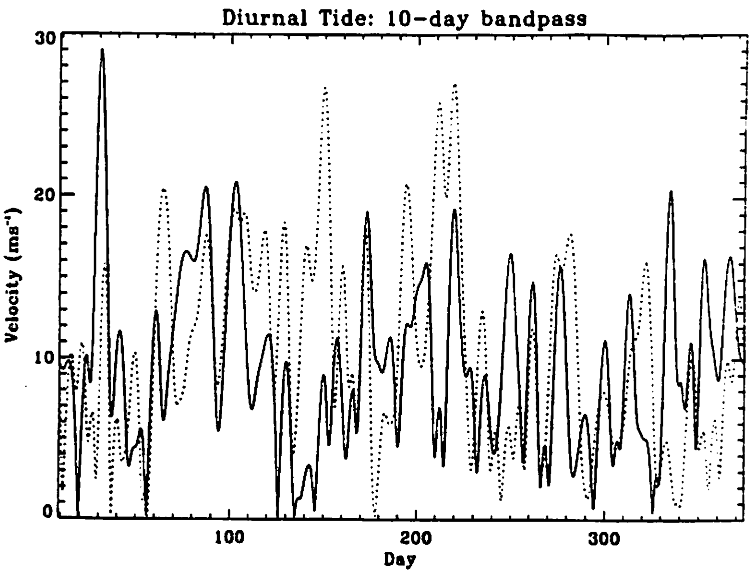
but carried out on all the data simultaneously.

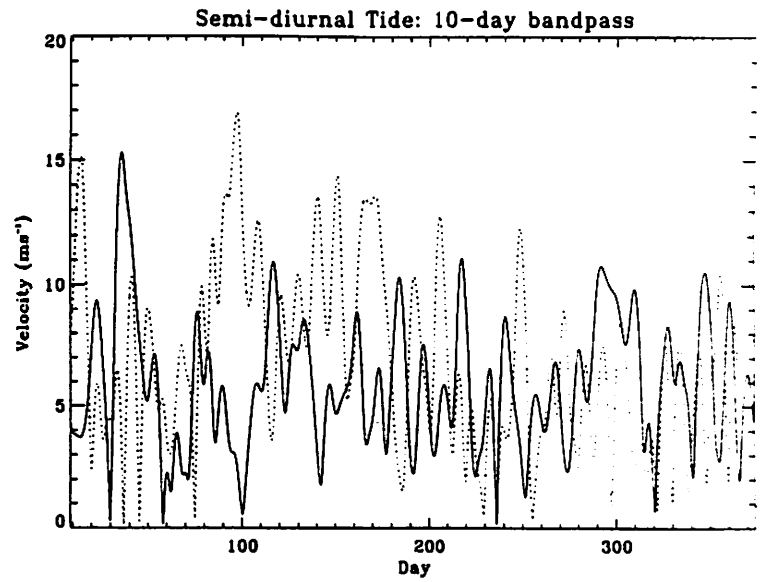


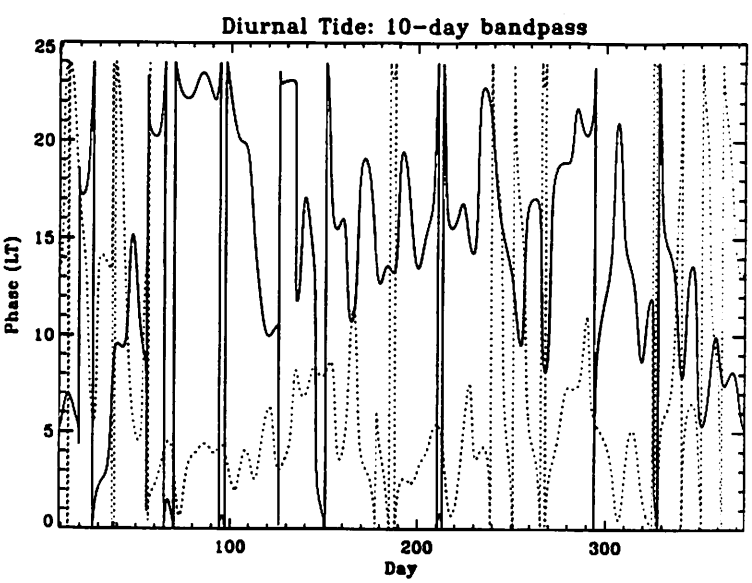


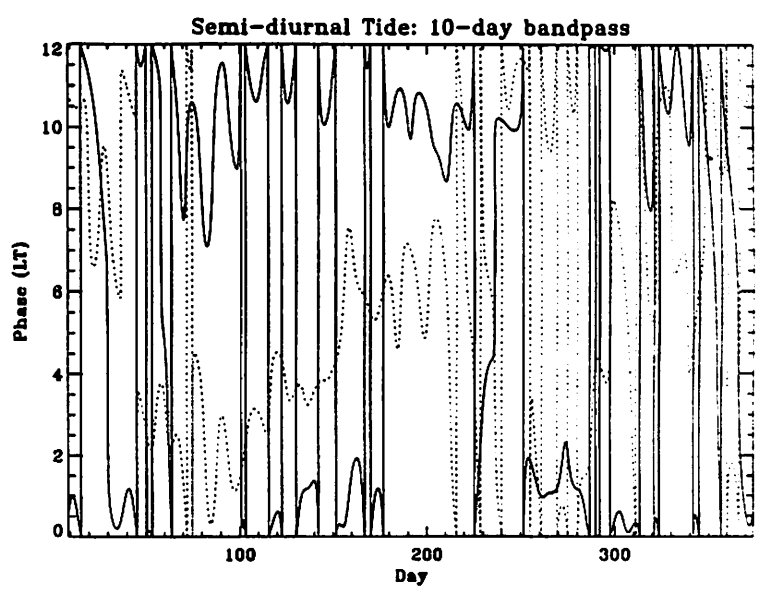




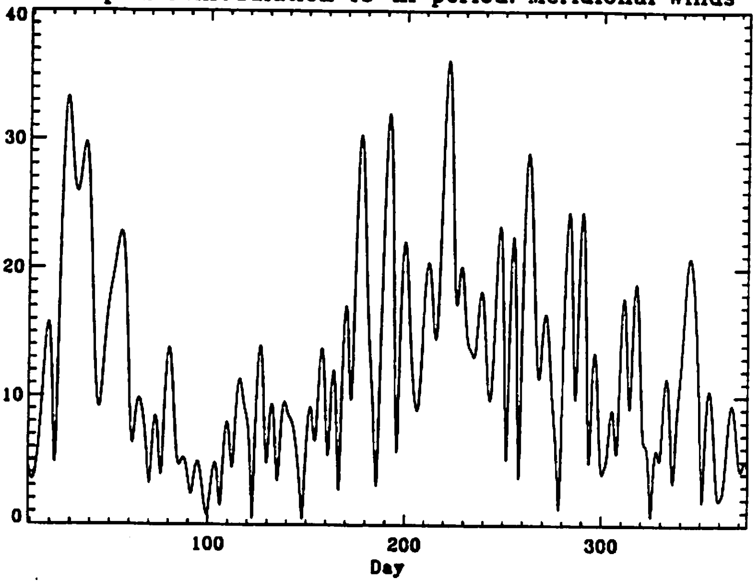

Diurnal Tide: 10-day bandpass
Velocity (ms⁻¹)
Day
0
10
20
30
100
200
300


Semi-diurnal Tide: 10-day bandpass
Velocity (ms⁻¹)
Day
0
5
10
15
20
100
200
300

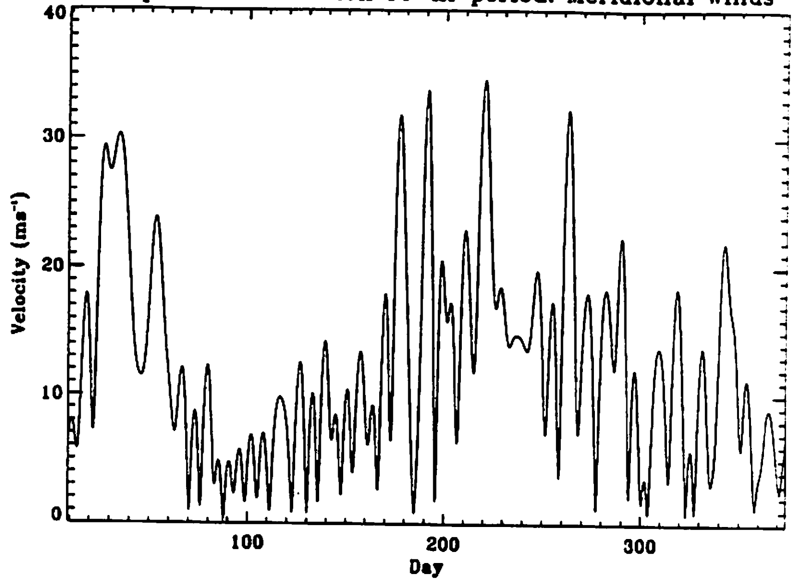

Diurnal Tide: 10-day bandpass
Phase (LT)
Day
0
5
10
15
20
25
100
200
300


Semi-diurnal Tide: 10-day bandpass
Phase (LT)
Day
0
2
4
6
8
10
12
100
200
300

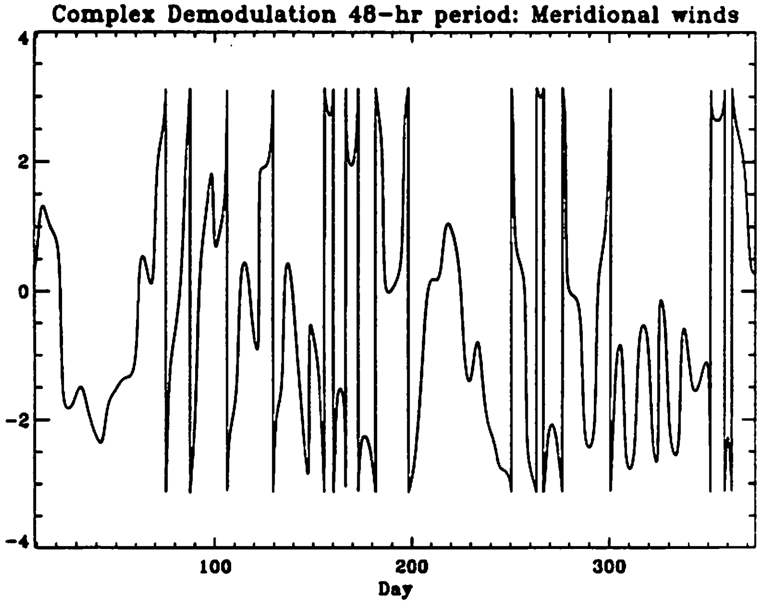
Complex Demodulation 48-hr period: Meridional winds



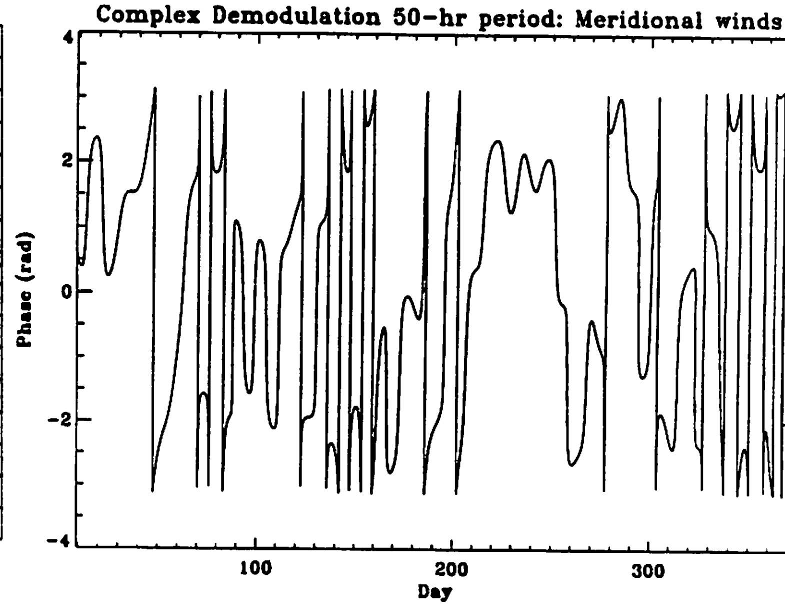
Complex Demodulation 50-hr period: Meridional winds

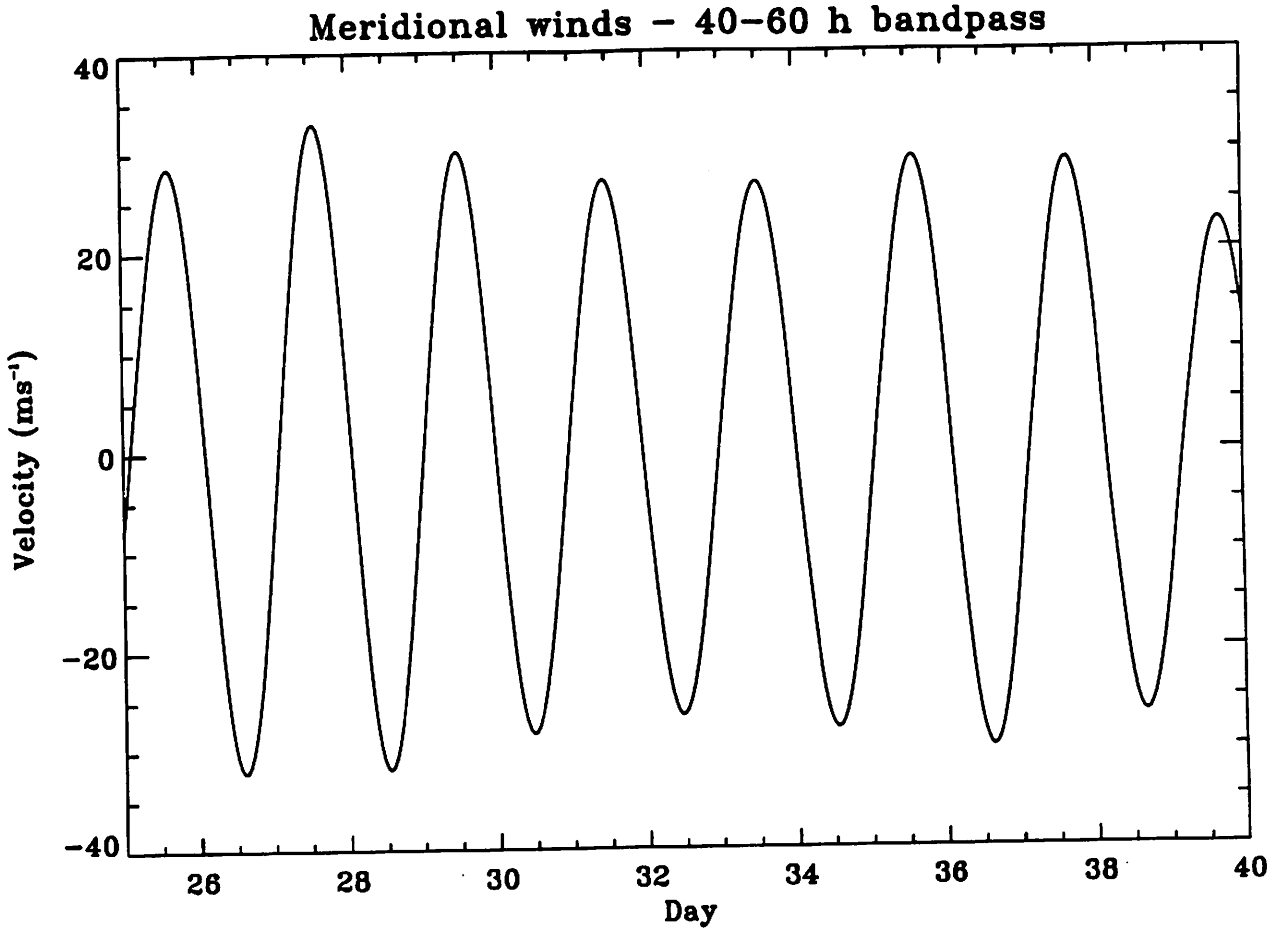


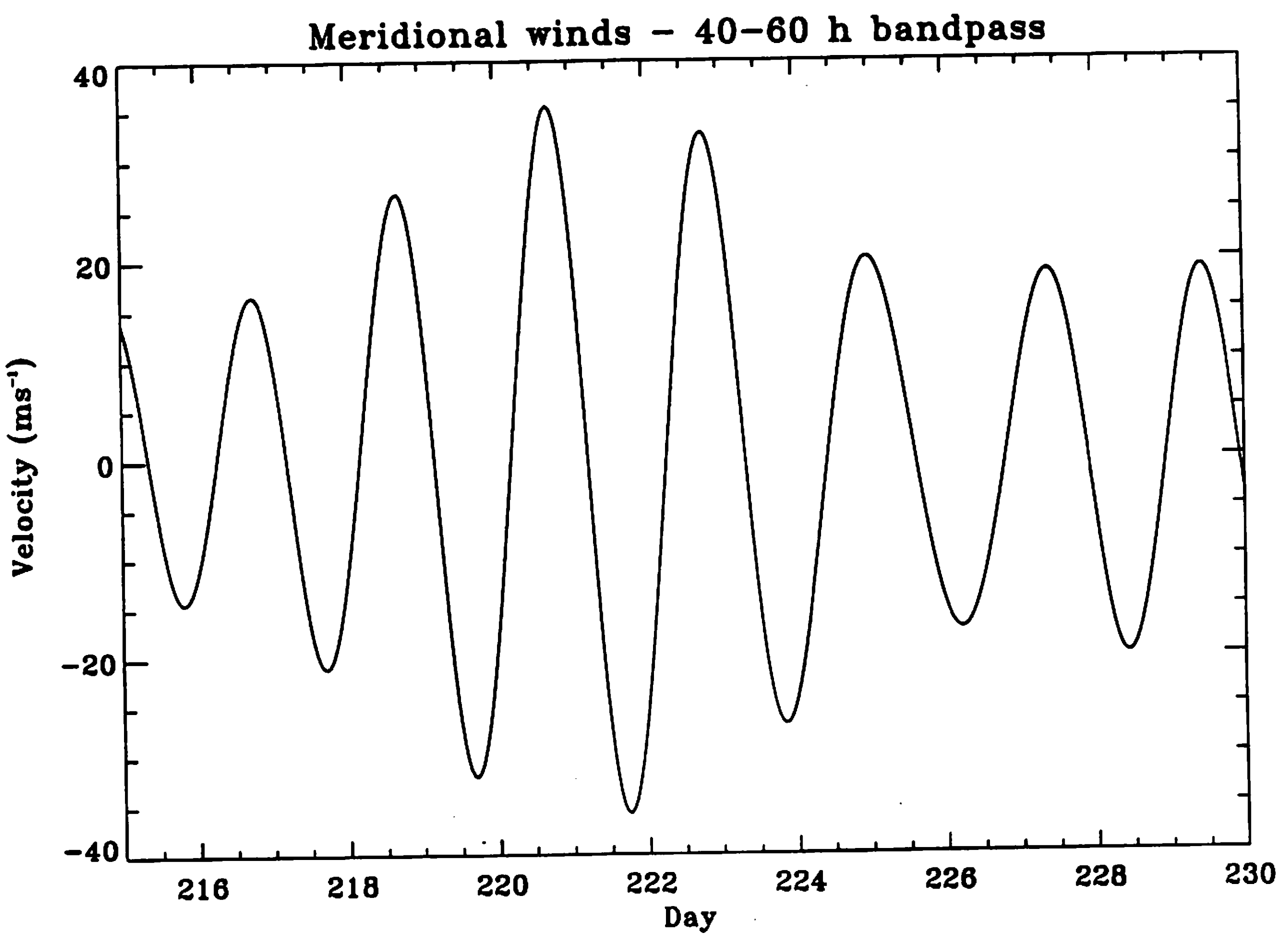
Complex Demodulation 48-hr period: Meridional winds



Complex Demodulation 50-hr period: Meridional winds




Meridional winds - 40-60 h bandpass
Velocity (ms⁻¹)
40
20
0
-20
-40
26
28
30
32
34
36
38
40
Day


Meridional winds - 40-60 h bandpass
Velocity (ms⁻¹)
40
20
0
-20
-40
216
218
220
222
224
226
228
230
Day

References

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