Introduction to the

FOURIER EQUATIONS

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TIME SERIES

CONTINUOUS ---- DISCRETE

DETERMINISTIC

If future values of a time series are determined by a formula or perscription it is said to be deterministic.

Non-probabilistic, generally properties can be computed analytically!

RANDOM

Stationary Non-Stationary

Generally described by mean, variance, and Autocorrelation or Spectral Density Function.

A. DETERMINISTIC SIGNALS

1. *PERIODIC SIGNALS*: Fourier Series - relation between a continuous periodic time function and a discrete frequency representation. (LINE SPECTRA)

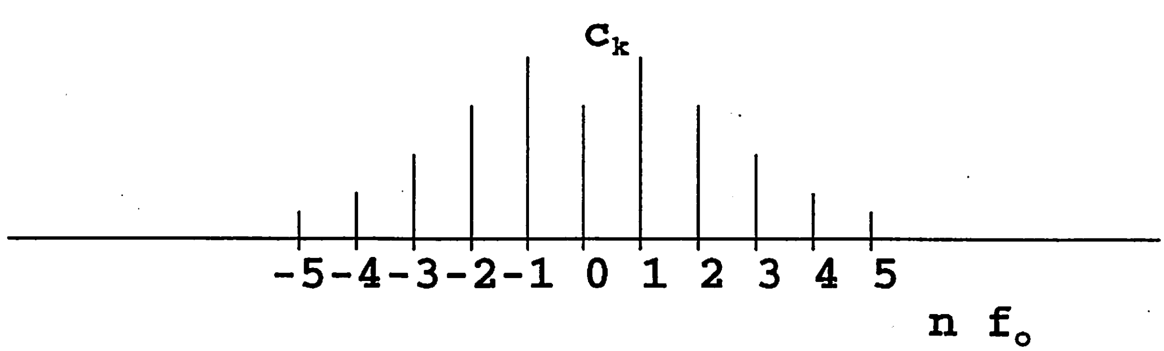
For a continuous periodic time signal with period T the complex form of the Fourier series is usually expressed as:

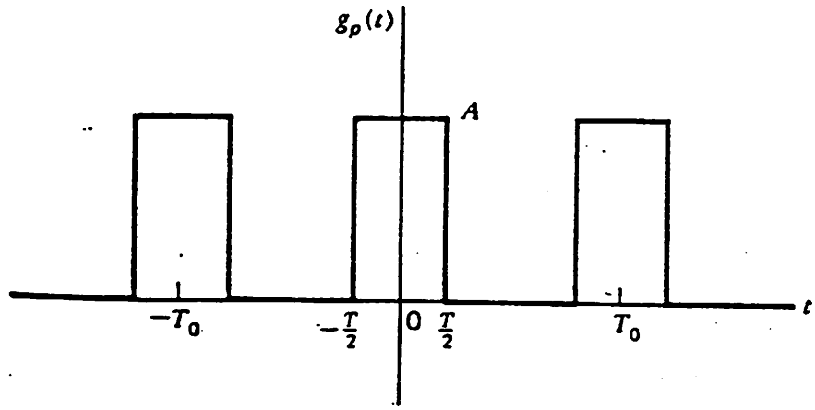
$$\tilde{S}_c(t) = \sum_{-\infty}^{+\infty} C_k e^{-j2\pi k f_o t}$$

where $f_o = 1/T$ and

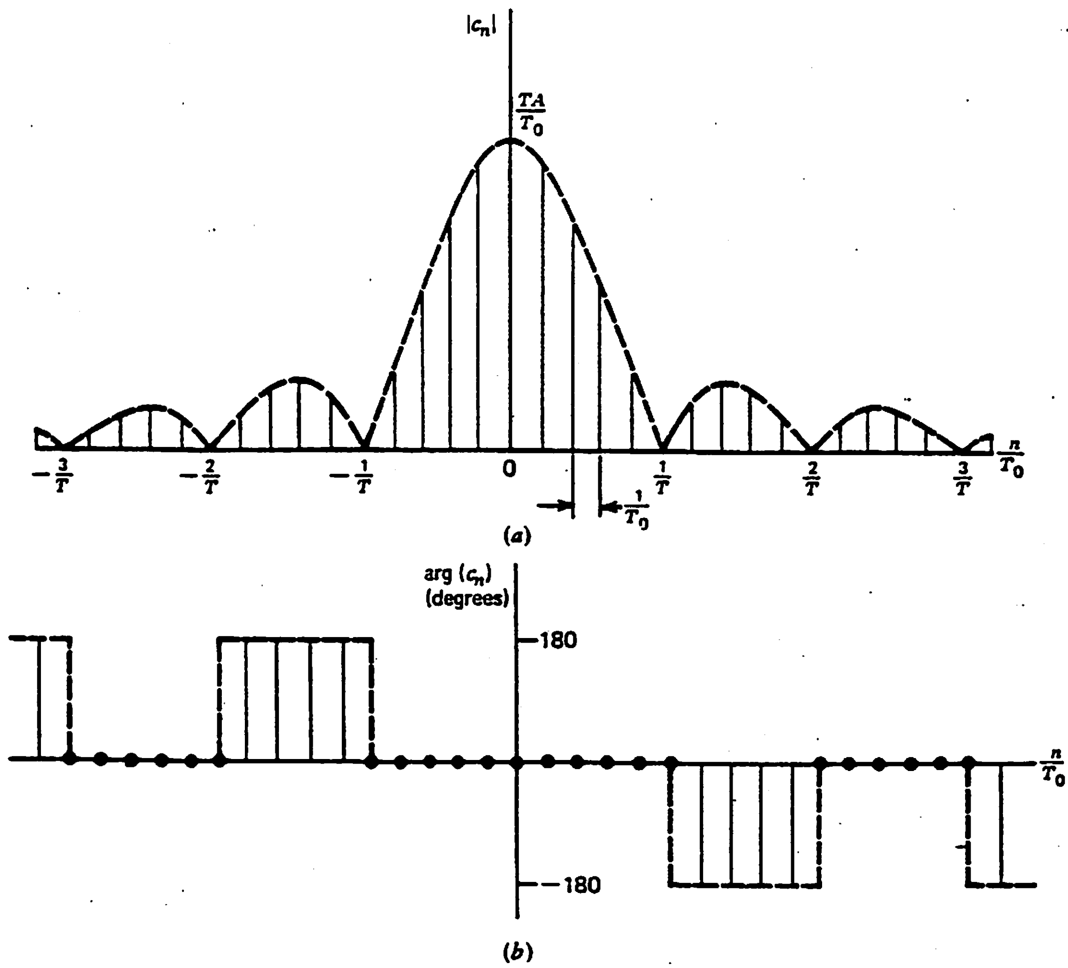
$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{S}_c(t)\, e^{-j2\pi k f_o t} dt$$

The c_k's are, in general, complex and can be thought of as the line amplitude spectrum for $x_c(t)$.





Periodic train of rectangular pulses of amplitude A, duration T, and period T_0.



Discrete spectrum of a periodic train of rectangular pulses for a duty cycle $T/T_0 = 0.2$. (a) Amplitude spectrum. (b) Phase spectrum.

Parseval's Theorem gives the total power as

$$P_T = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |s_c(t)|^2 dt = \sum_{-\infty}^{\infty} |c_k|^2$$

Thus, we could call $|c_k|^2$ the Power Spectral Density.

2. *NON-PERIODIC SIGNALS*:

In this case the continuous Fourier Transform relates the time signal and its frequency representation.

$$S_c(f) = \int_{-\infty}^{\infty} s_c(t)\, e^{-j2\pi ft}\, dt$$

and

$$s_c(t) = \int_{-\infty}^{+\infty} S_c(f)\, e^{j2\pi ft}\, df$$

$S_c(f)$ is in general complex and for real signals has a real part that is an even function and an imaginary part that is an odd function.

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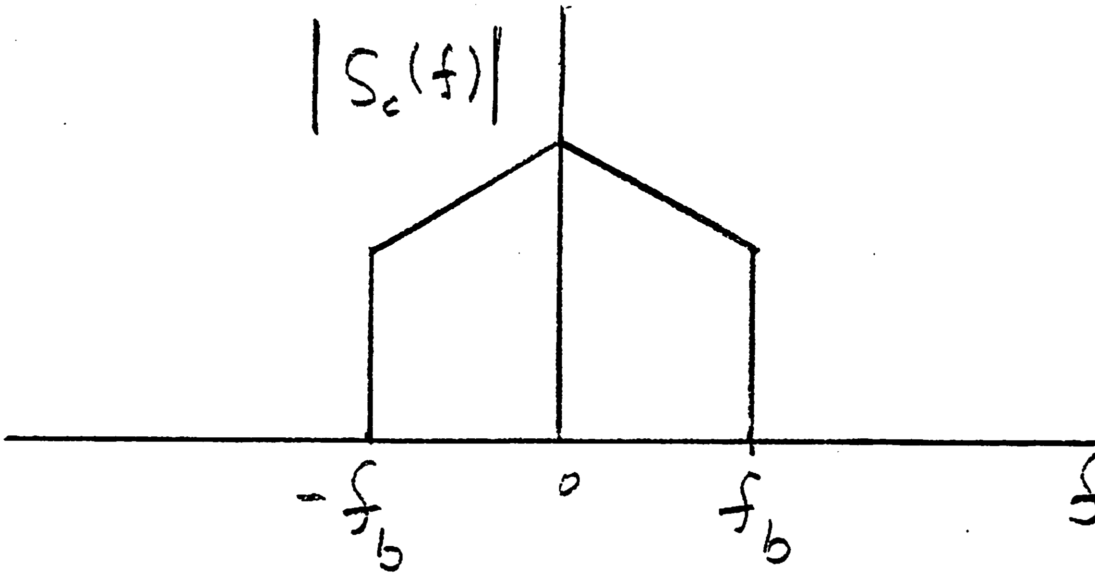

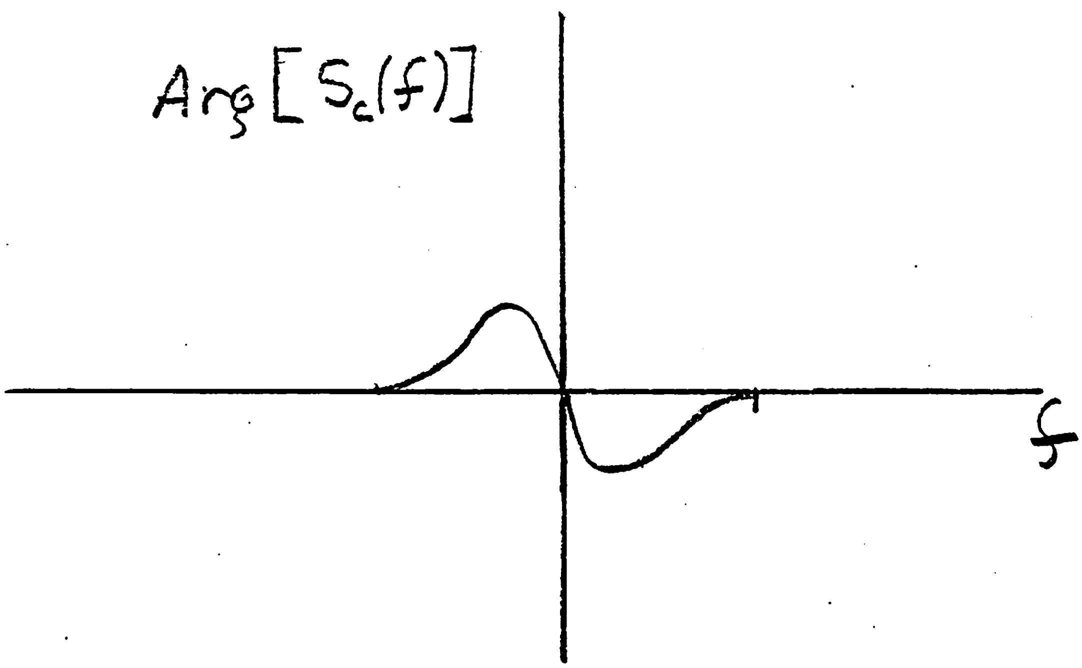

f_b = bandwidth

WARNING -- WARNING -- WARNING -- WARNING

The Fourier Transform equations may be defined differently by different authors. Some common definitions are:

$$S_c(f) = \int_{-\infty}^{\infty} s_c(t)\, e^{-j2\pi ft}\, dt$$

$$s_c(t) = \int_{-\infty}^{\infty} S_c(f)\, e^{j2\pi ft}\, df$$

or

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

or

$$F(j\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

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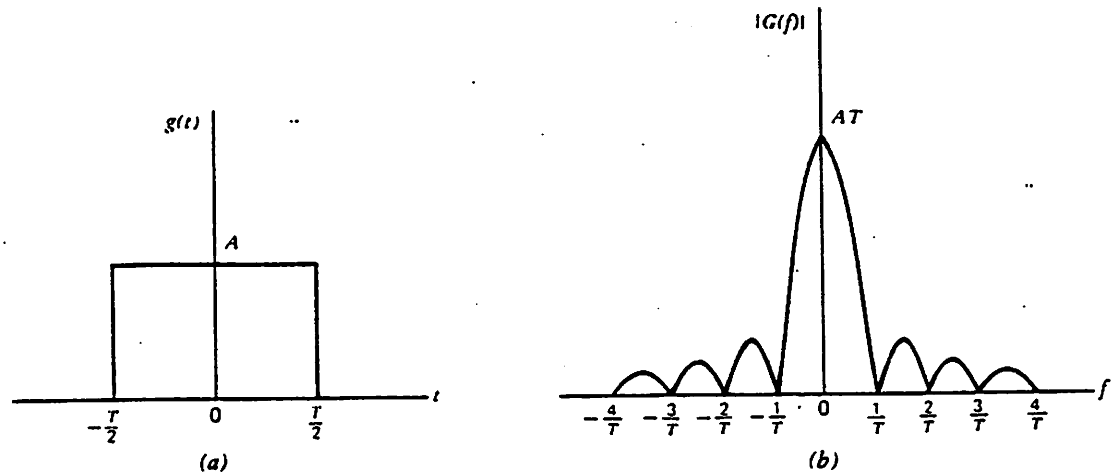
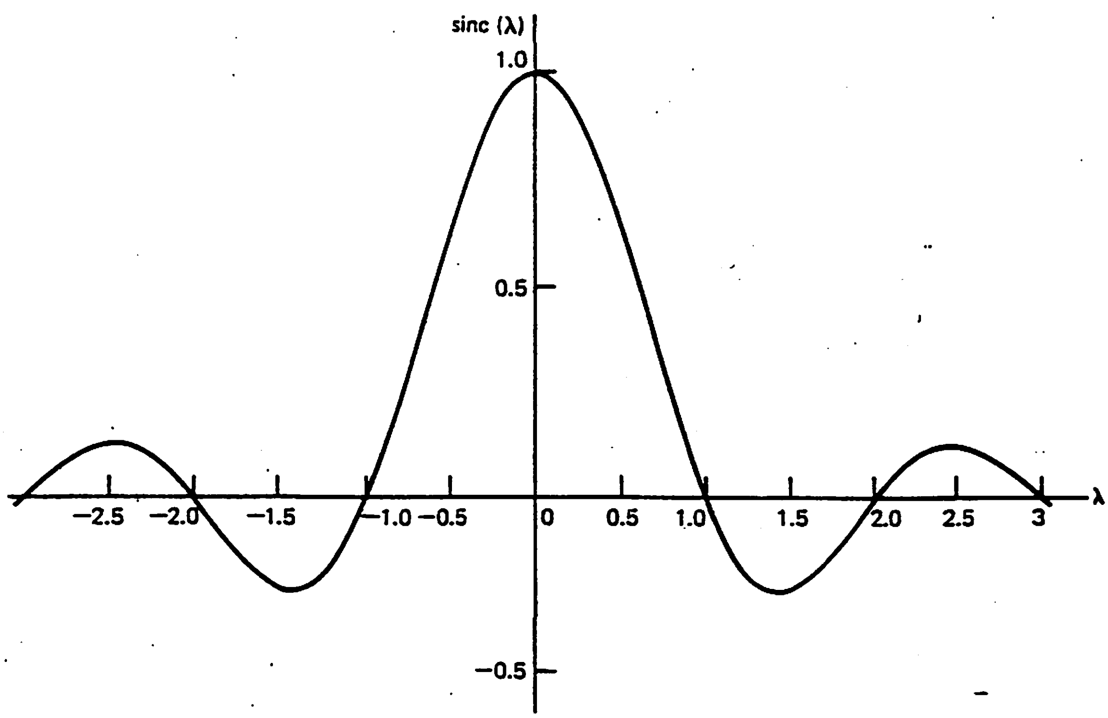
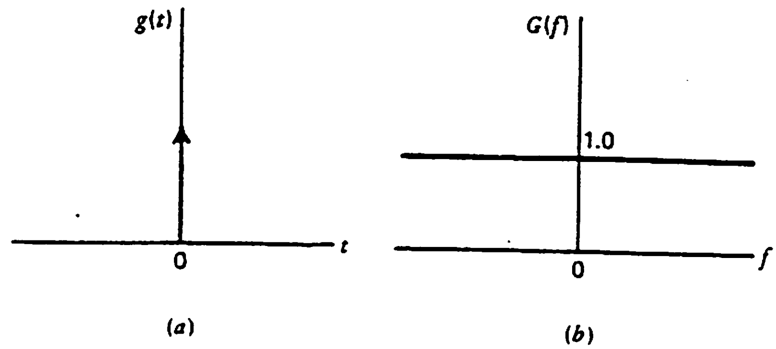


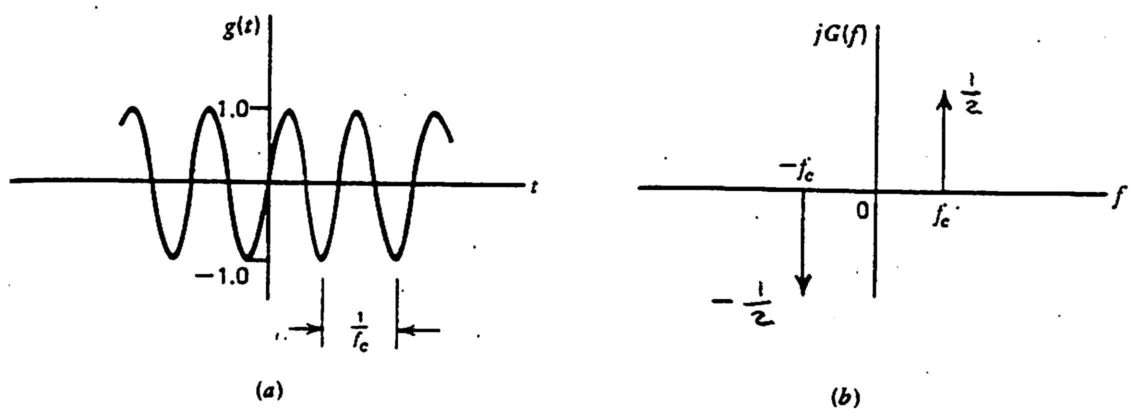
(a) Rectangular pulse. *(b)* Amplitude spectrum.



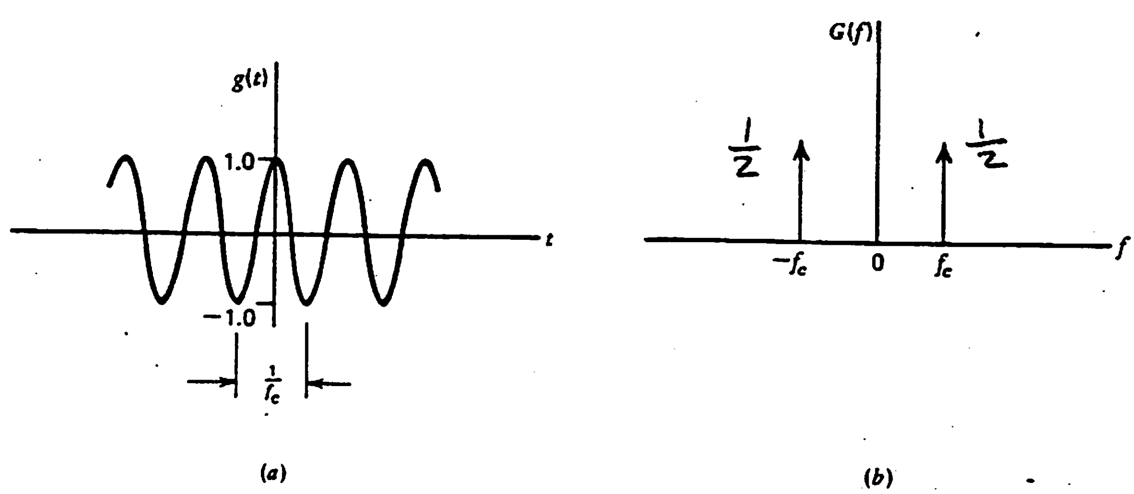
The sinc function.



(a) Dirac delta function $\delta(t)$. (b) Spectrum of $\delta(t)$.



(a) Sine function. (b) Spectrum.



(a) Cosine function. (b) Spectrum.

Properties of the Fourier Tranform:

a. Convolution

$$s_1(t) \otimes s_2(t) \Leftrightarrow S_1(f)\ S_2(f)$$

$$s_3(t) = \int_{-\infty}^{\infty} s_1(\tau)\, s_2(t-\tau)\, d\tau \Leftrightarrow S_1(f)\ S_2(f)$$

b. Rayleigh's Energy Theorem

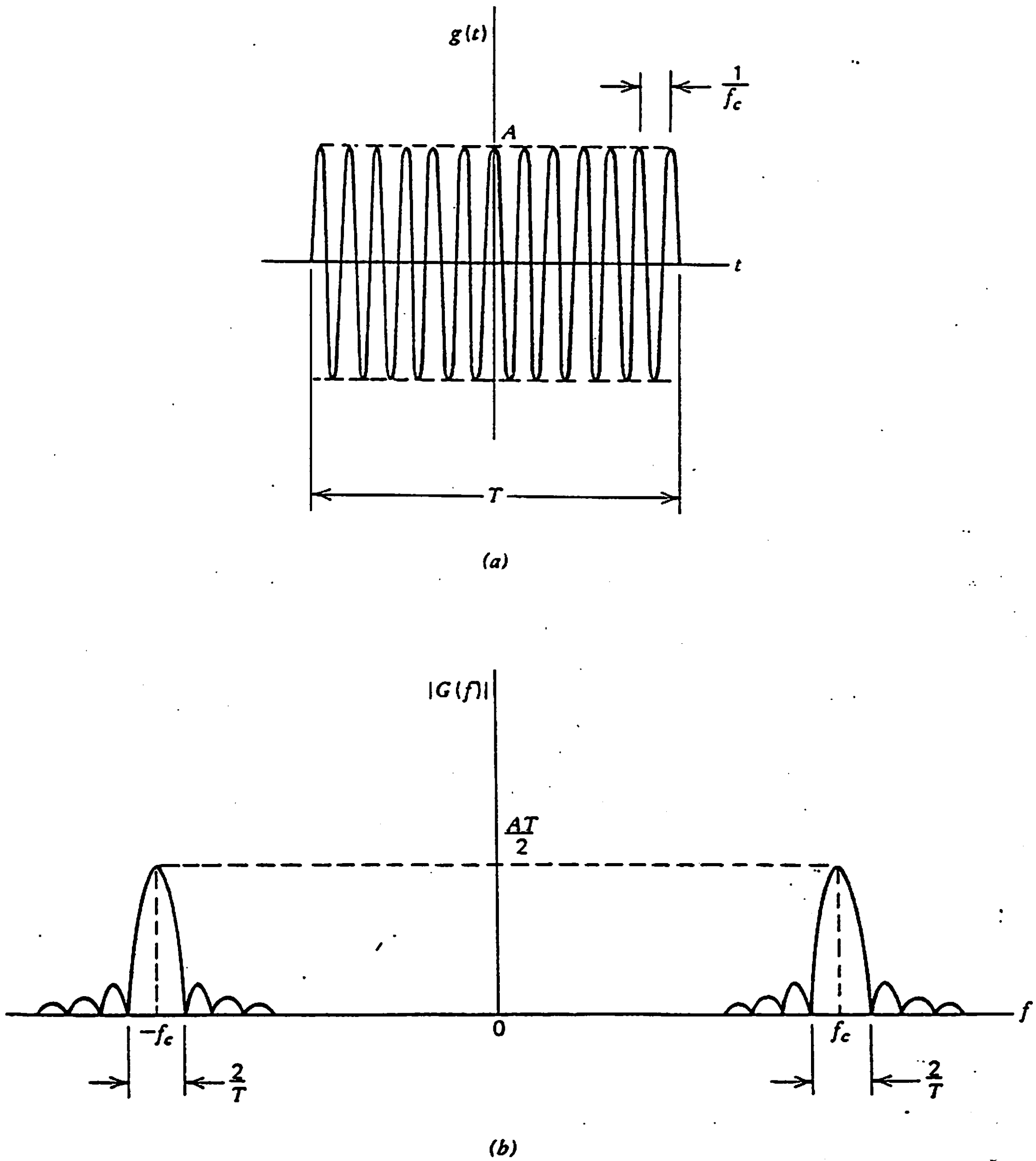
$$E_T = \int_{-\infty}^{\infty} |s_c(t)|^2\, dt = \int_{-\infty}^{\infty} |S_c(f)|^2\, df$$

and the Energy over a frequency range 0 to Δf

$$E = \int_{-\Delta f}^{\Delta f} |S_c(f)|^2\, df$$

so: $|S_c(f)|^2$ looks like an

Energy Spectral Density.



(a) RF pulse. *(b)* Amplitude spectrum.

CORRELATION

$$R_{u,v}(\tau) = \int_{-\infty}^{\infty} u_c(t)\ v_c(t-\tau)\ dt$$

for energy signals, and

$$R_{u,v}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_c(t)\ v_c(t-\tau)\ dt$$

for Power Signals.

If $u = v$ -- autocorrelation
$u \neq v$ -- crosscorrelation

By expressing correlation as a convolution one can show that

$$R_s(\tau) \Leftrightarrow |S_c(f)|^2$$

So:

$$\begin{array}{ccc} s_c(t) & \overset{F}{\Leftrightarrow} & S_c(f) \\ \text{AC} \Downarrow & & \Downarrow |\ |^2 \\ R_s(\tau) & \overset{F}{\Leftrightarrow} & P_s(f) \end{array}$$

RELATIONSHIP BETWEEN FOURIER SERIES AND FOURIER TRANSFORM

If $s_c(t)$ = one period of $\tilde{s}_c(t)$ then

$$P_T=\frac{1}{T}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} |s_c(t)|^2 dt=\sum_{k=-\infty}^{\infty} |c_k|^2=\frac{1}{T_o^2}\sum_{n=-\infty}^{\infty} |S_c(\frac{n}{T_o})|^2$$

or

$$c_k = \frac{1}{T_o} S_c(f)\Big|_{f=\frac{k}{T_o}}$$

Thus, under the stated conditions, to within a constant, the Fourier Series complex coefficients represent samples of the Fourier Transform at $f = k / T_o$.

B. RANDOM SIGNALS

Each particular random signal is assumed to be one of an ensemble of sample signals.

If one treats each sample signal as a deterministic signal, a computed Fourier Transform or Power Spectrum is not a very good estimate of the 'real' Transform or Spectrum of the statistical process!!??

However, for a stationary random ergodic process the autocorrelation function is a deterministic function and can be found by

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t-\tau)\ x(t)\ dt$$

$$\textit{Variance of } R_x(\tau) \to 0 \textit{ as } T \to \infty$$

Called an unbiased estimate. Then

$$R_x(\tau) \quad \Leftrightarrow \quad P_x(f)$$

(i.e., the Autocorrelation function of the random process X and the Power Spectral density function of the process are Fourier Transform pairs.)

C. DISCRETE TIME SIGNALS

1. PERIODIC SIGNALS: We now deal with the Discrete Fourier Series (DFS). For a discrete time signal with period T we have

$$\tilde{x}[n\Delta t] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[\frac{k}{T}]\, e^{j2\pi \frac{k}{T} n\Delta t} \qquad T = N\Delta t$$

$$\tilde{X}[\frac{k}{T}] = \sum_{n=0}^{N-1} \tilde{x}[n\Delta t]\, e^{-j2\pi \frac{k}{T} n\Delta t}$$

sometimes the definition

$$W_N = e^{-j(\frac{2\pi}{N})} \quad \textit{is used, then } e^{-j2\pi \frac{k}{T} n\Delta t} = W_N^{nk}$$

NOTE THAT:

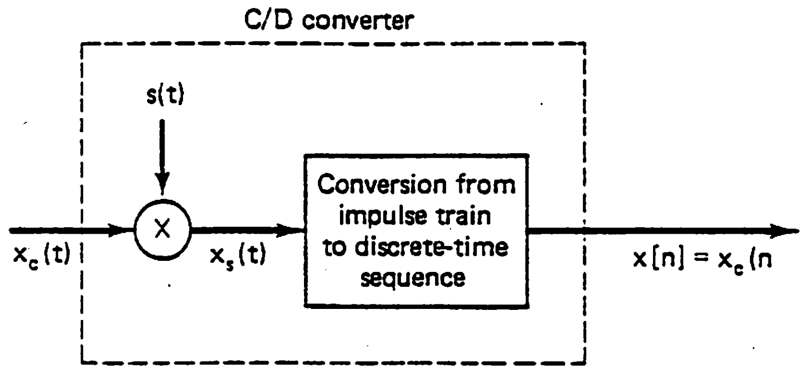
A DISCRETE and PERIODIC function in TIME -- results in -- A DISCRETE and PERIODIC function in FREQUENCY

2. NON-PERIODIC DISCRETE TIME SIGNALS: We now have the Discrete Time Fourier Transform (DTFT).

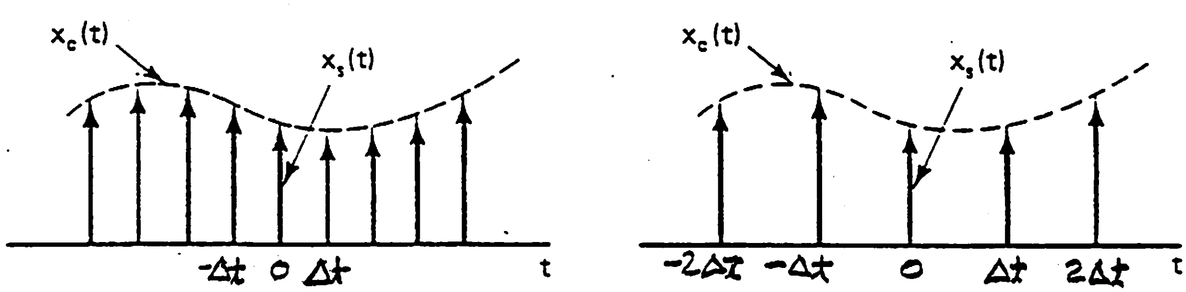
$$X(f) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\, e^{-j2\pi f n\Delta t} \qquad f_s = \frac{1}{\Delta t}$$

$$x(n\Delta t) = \Delta t \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f)\, e^{j2\pi f n\Delta t}\, df$$

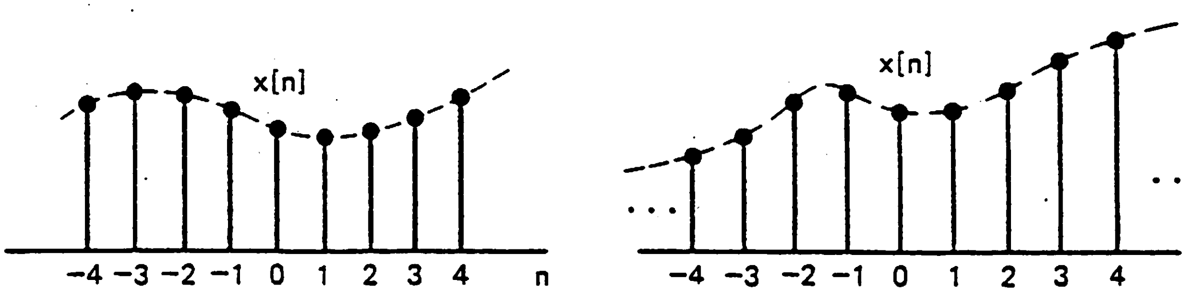
X(f) will be CONTINUOUS and PERIODIC in $f_s = 1/\Delta t$


C/D converter
s(t)
Conversion from impulse train to discrete-time sequence
$x_c(t)$
$x_s(t)$
$x[n] = x_c(n$

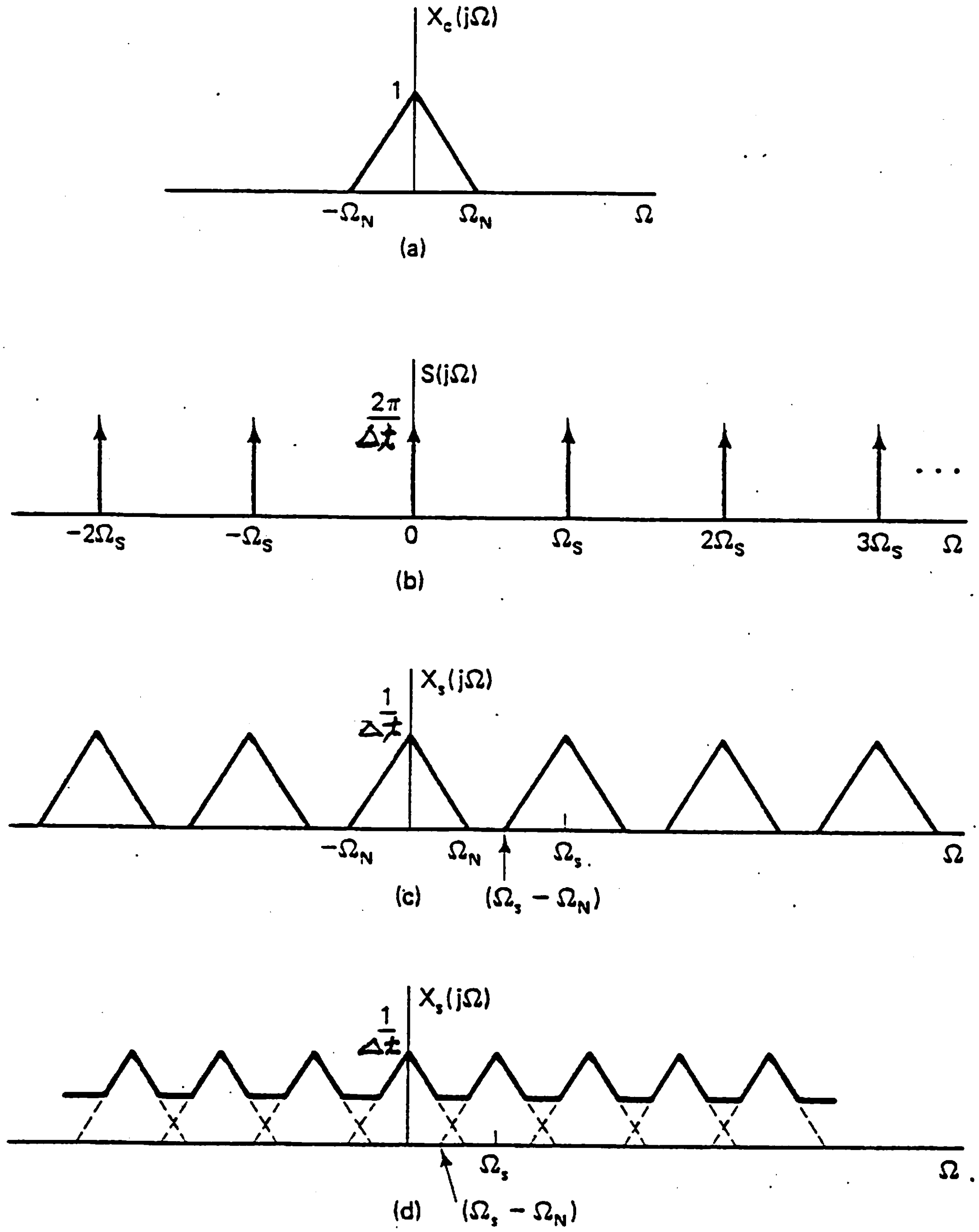
(a)


$x_c(t)$
$x_s(t)$
$-\Delta t$ 0 Δt
t
$x_c(t)$
$x_s(t)$
$-2\Delta t$ $-\Delta t$ 0 Δt $2\Delta t$
t

(b)


x[n]
−4 −3 −2 −1 0 1 2 3 4
n
x[n]
−4 −3 −2 −1 0 1 2 3 4

(c)



$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$S(j\Omega)$
$\frac{2\pi}{\Delta t}$
$-2\Omega_S$
$-\Omega_S$
0
Ω_S
$2\Omega_S$
$3\Omega_S$
Ω
(b)
$X_s(j\Omega)$
$\frac{1}{\Delta t}$
$-\Omega_N$
Ω_N
Ω_s
Ω
$(\Omega_s - \Omega_N)$
(c)
$X_s(j\Omega)$
$\frac{1}{\Delta t}$
Ω_s
Ω
(d)
$(\Omega_s - \Omega_N)$

3. FINITE-DURATION DISCRETE TIME SIGNALS: Here we define the Discrete Fourier Transform (DFT) as the first N terms of the Discrete Fourier Series Coefficients X[k/T] or

$$X[\frac{k}{T}] = \sum_{n=0}^{N-1} x[n\,\Delta t]\, e^{-j\frac{2\pi}{N}kn}$$

$$x[n\,\Delta t] = \frac{1}{N}\sum_{k=0}^{N-1} X[\frac{k}{T}]\, e^{j\frac{2\pi}{N}kn} \qquad 0 \le n \le N-1$$

or for $\Delta t = 1$ and dropping the 1/T in the frequency index the DFT is normally written as

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

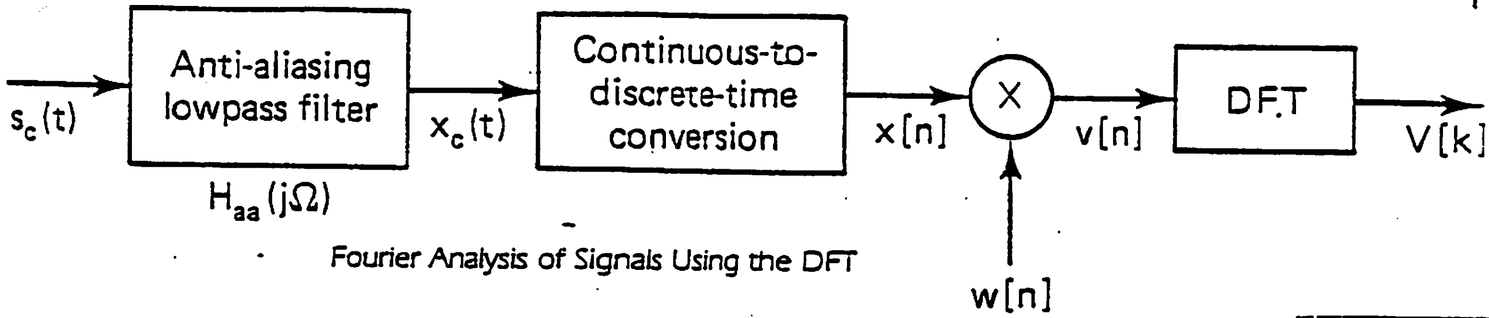
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\; W_N^{-kn}$$

So the DFT samples the DTFT at N points in frequency 1/T apart over the frequency range

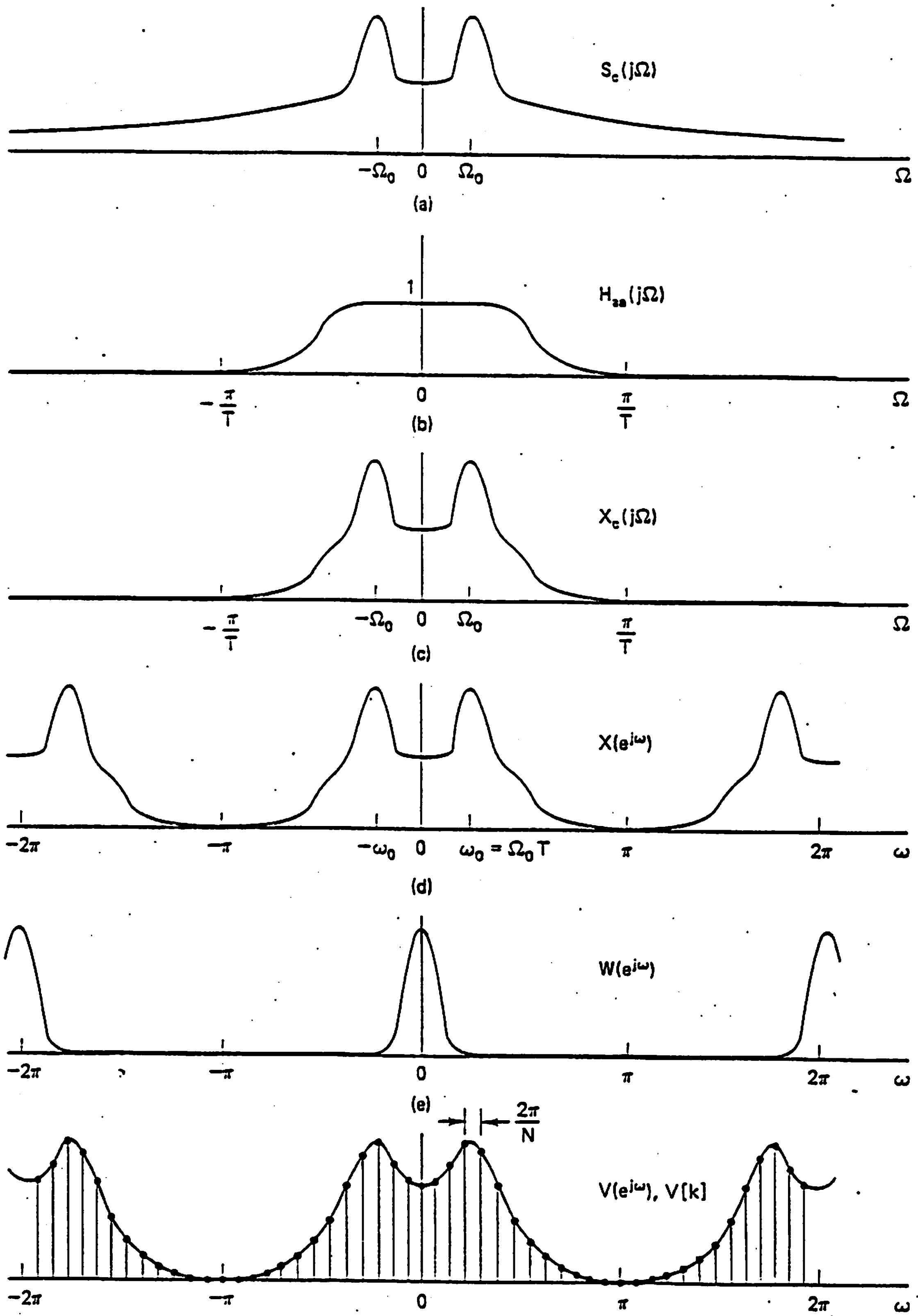
$$-\frac{f_s}{2} \le f \le \frac{f_s}{2}$$

or

$$X[k] = X(f)\,|_{f=\frac{k}{T}}$$



Fourier Analysis of Signals Using the DFT

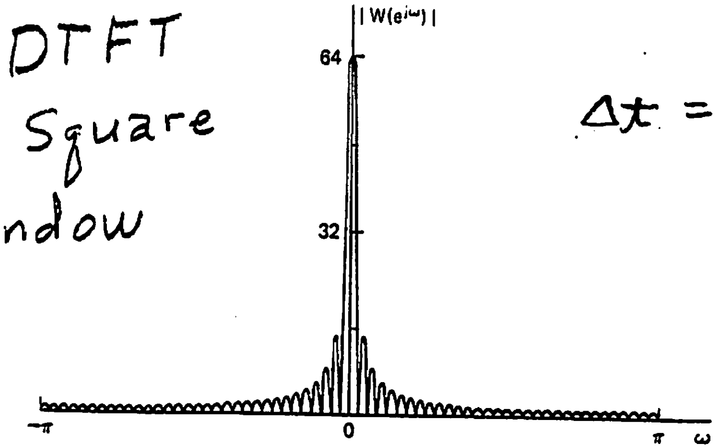


N K

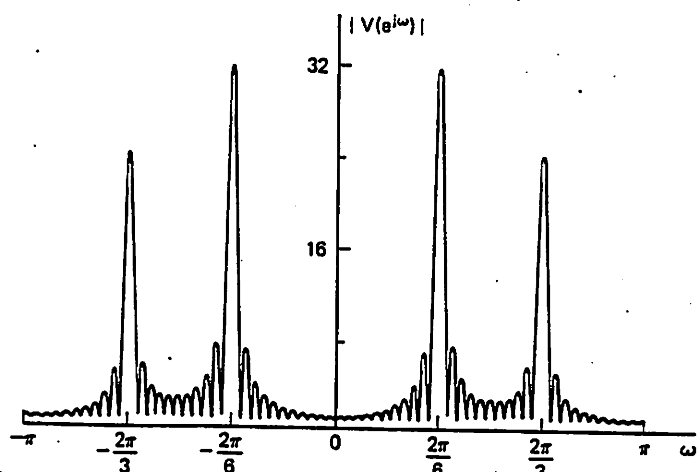
N/T f (actual)

DTFT of Square Window

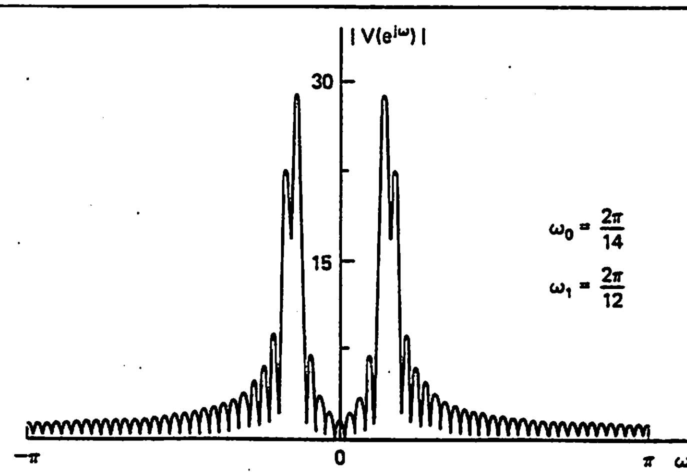
$\Delta t = 1$



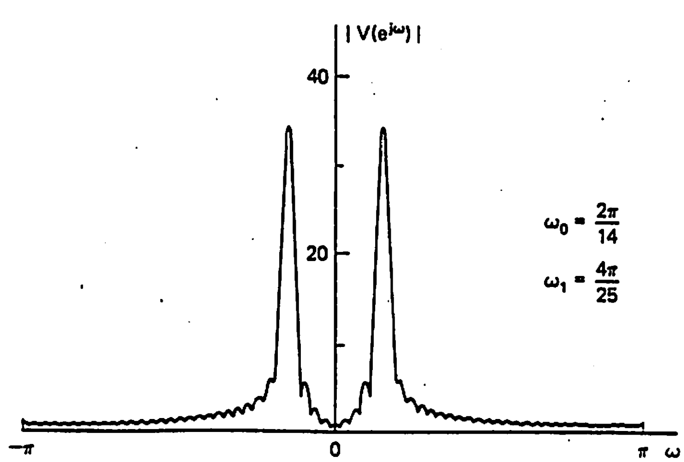
(a)



(b)

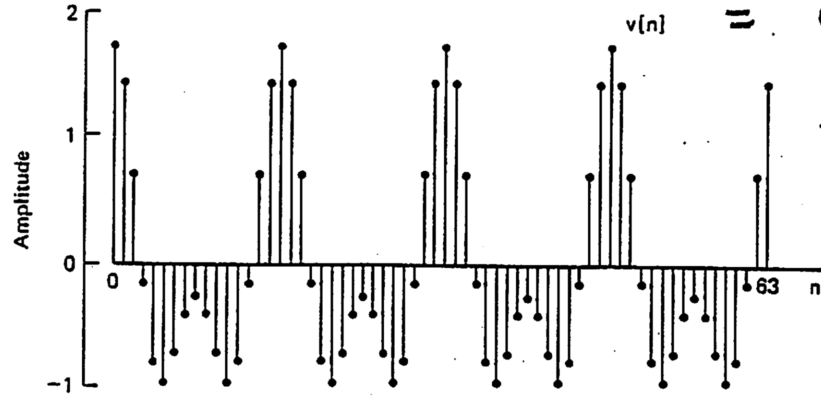


(d)

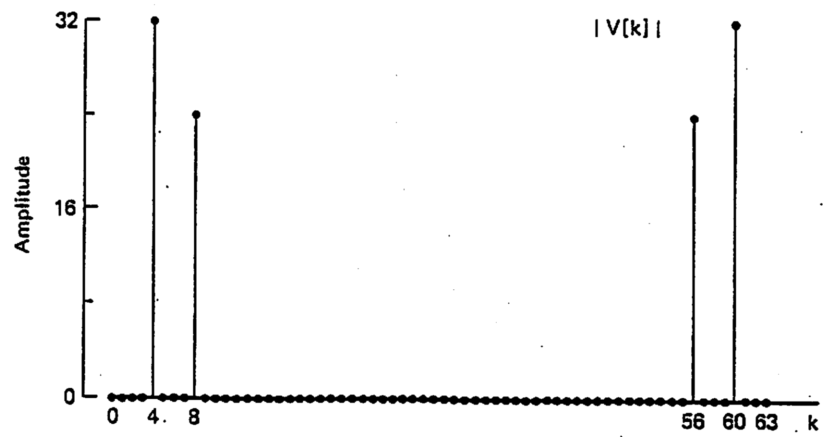


2

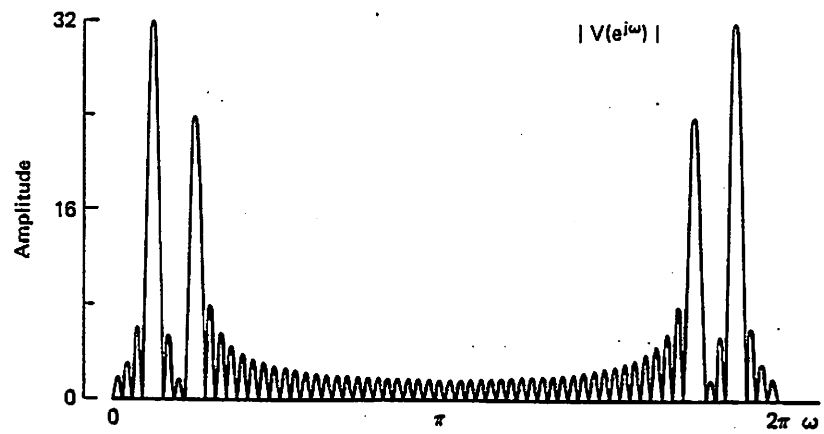
$= \cos \frac{2\pi}{16} n + .75 \cos \frac{2\pi}{8} n$



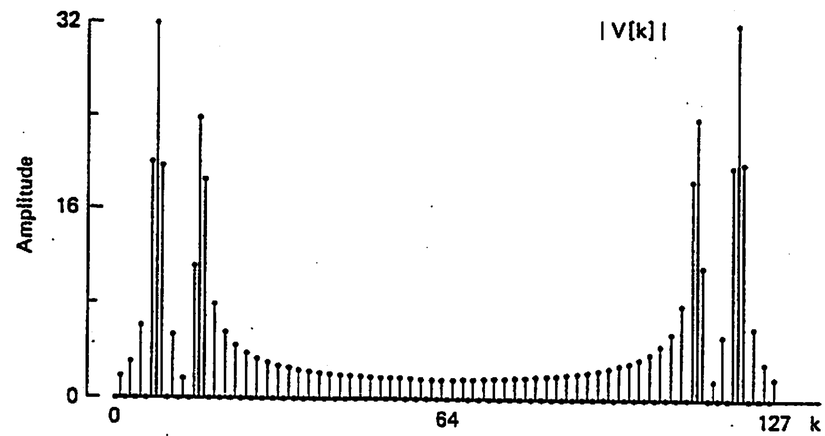
(a)

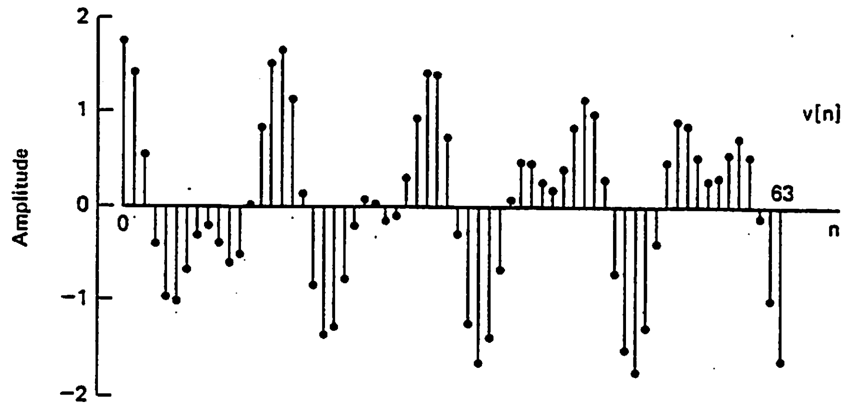


(b)



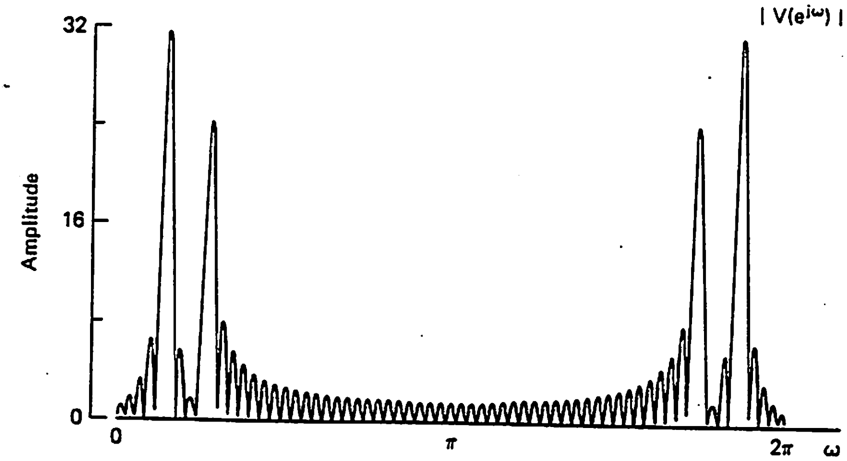
(c)



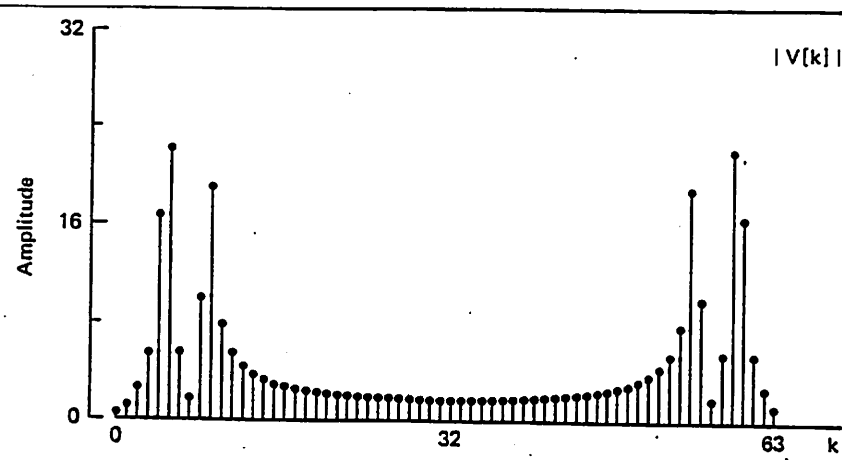


(a)

$v[n] = \cos \frac{2\pi}{14} n + .75 \cos \frac{4\pi}{15} n$

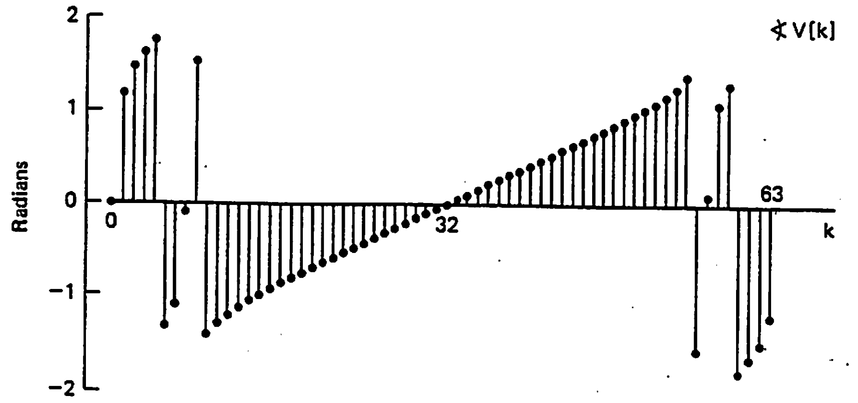


DTFT
$\Delta t = 1$



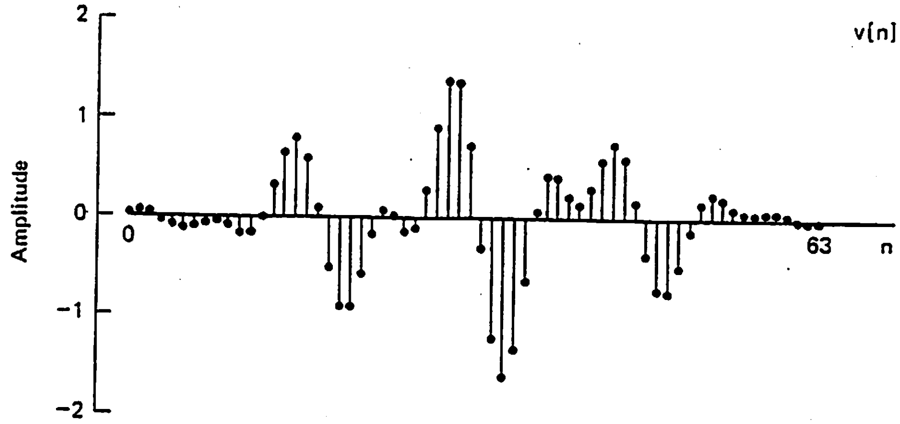
(d)

DFT

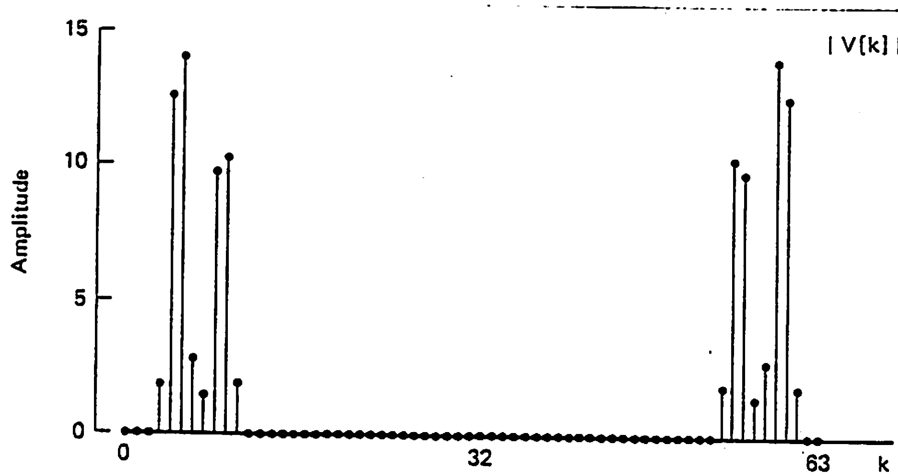


(e)

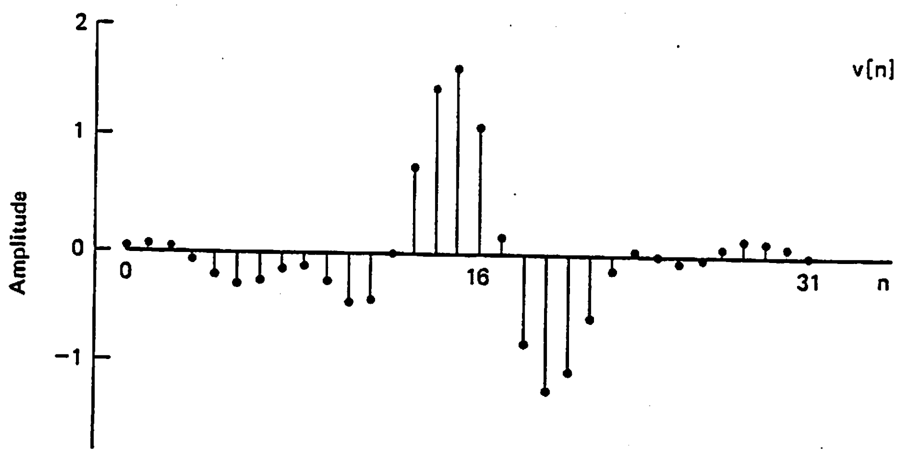
Kaiser Window



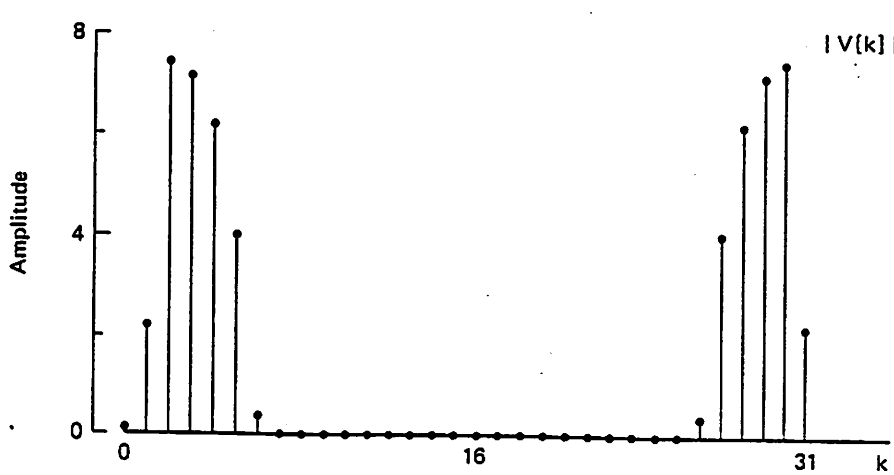
(a)



(b)



(c)



(d)

LOOKING BACK OVER WHAT WE HAVE TALKED ABOUT ONE CAN SEE OR DERIVE THE FOLLOWING RESULTS

$$X[k] = N c_k = X(f)\big|_{f=\frac{k}{T}} = \frac{1}{\Delta t} S_c(f)\big|_{f=\frac{k}{T}} \quad \textit{where } T = N\Delta t$$

NOW, WHAT ABOUT POWER OR ENERGY SPECTRA?

FOR A CONTINUOUS PERIODIC SIGNAL --

THE MAGNITUDE SQUARED FOURIER SERIES COEFFICIENT COULD BE VIEWED AS A LINE (OR DISCRETE) POWER SPECTRUM. SO

$$P(f)\big|_{f=\frac{k}{T}} = |c_k|^2 \textit{ can be approximated by } \frac{|X[k]|^2}{N^2}$$

FOR A CONTINUOUS TIME-LIMITED SIGNAL --

THE MAGNITUDE SQUARED CONTINUOUS FOURIER TRANSFORM TURNED OUT TO BE THE ENERGY DENSITY SPECTRUM. SO

$$P(f)\big|_{BW\,\Delta f} = \int_{BW\,\Delta f} \frac{|S_c(f)|^2}{T} df \simeq \frac{(\Delta t)^2 |X[k]|^2 \Delta f}{T}$$

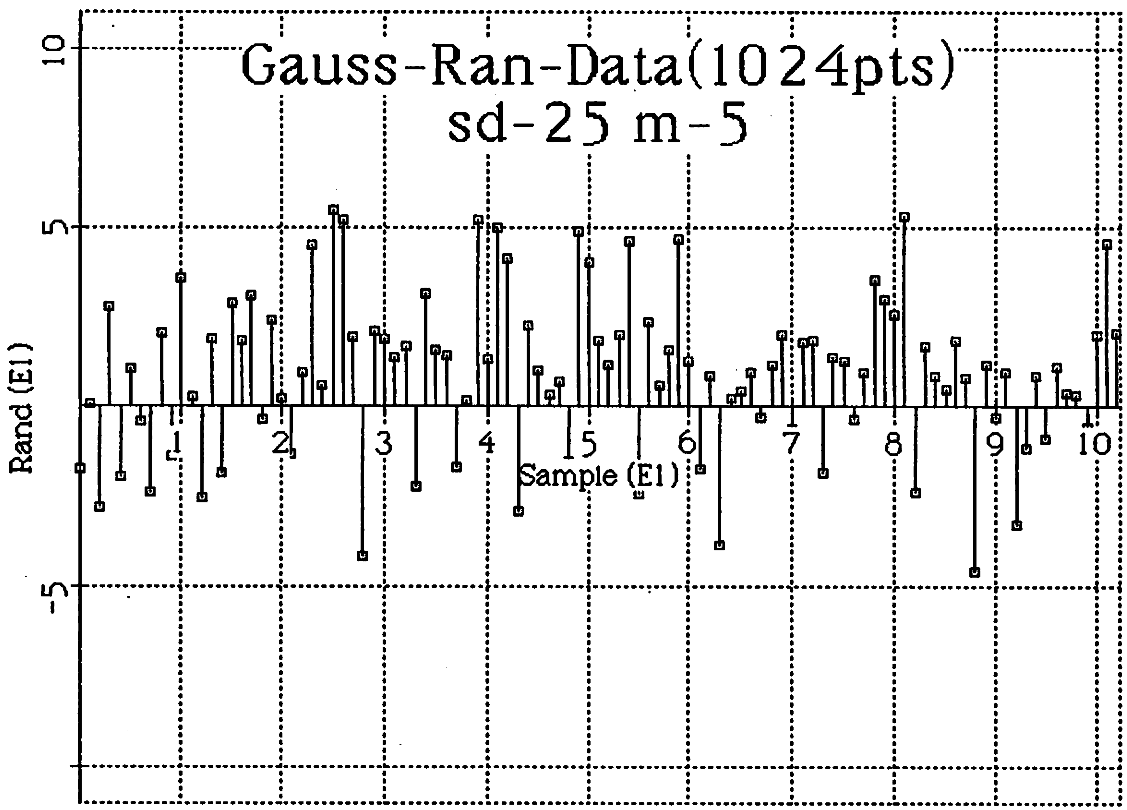
or

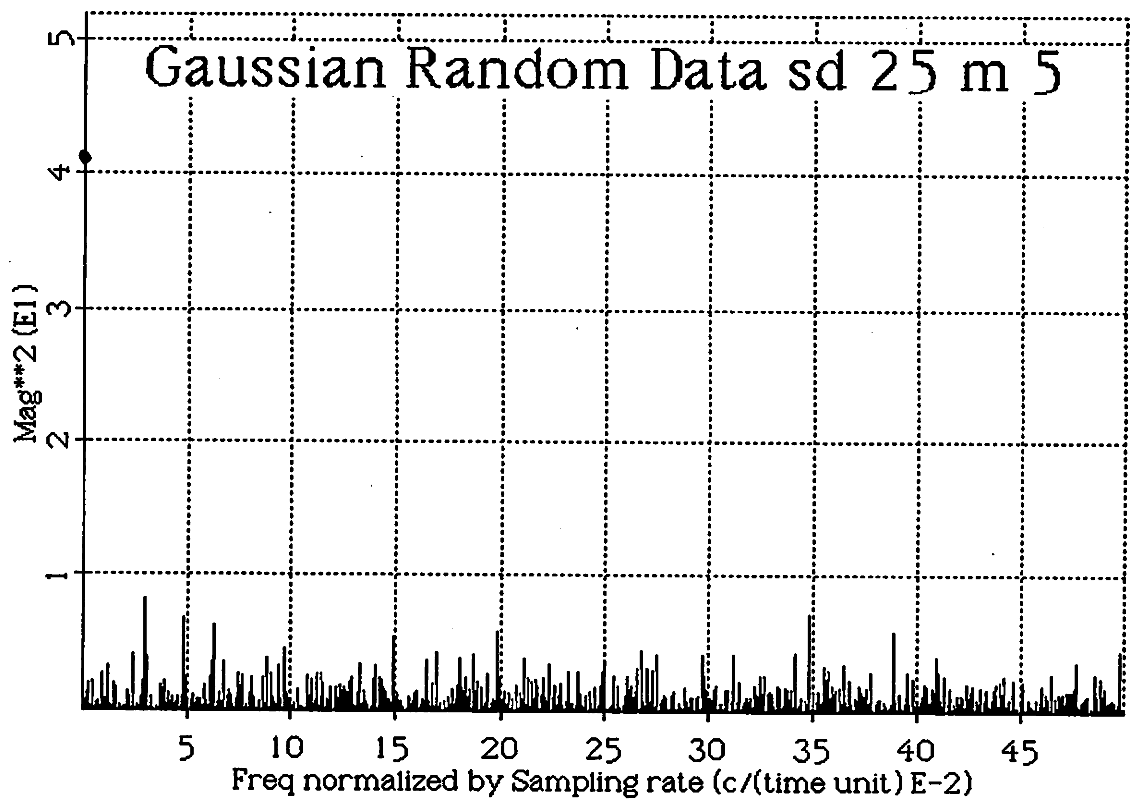
$$\frac{|X[k]|^2}{N^2} \quad \textit{where again } \Delta f = \frac{1}{T} = \frac{1}{N\Delta t}$$

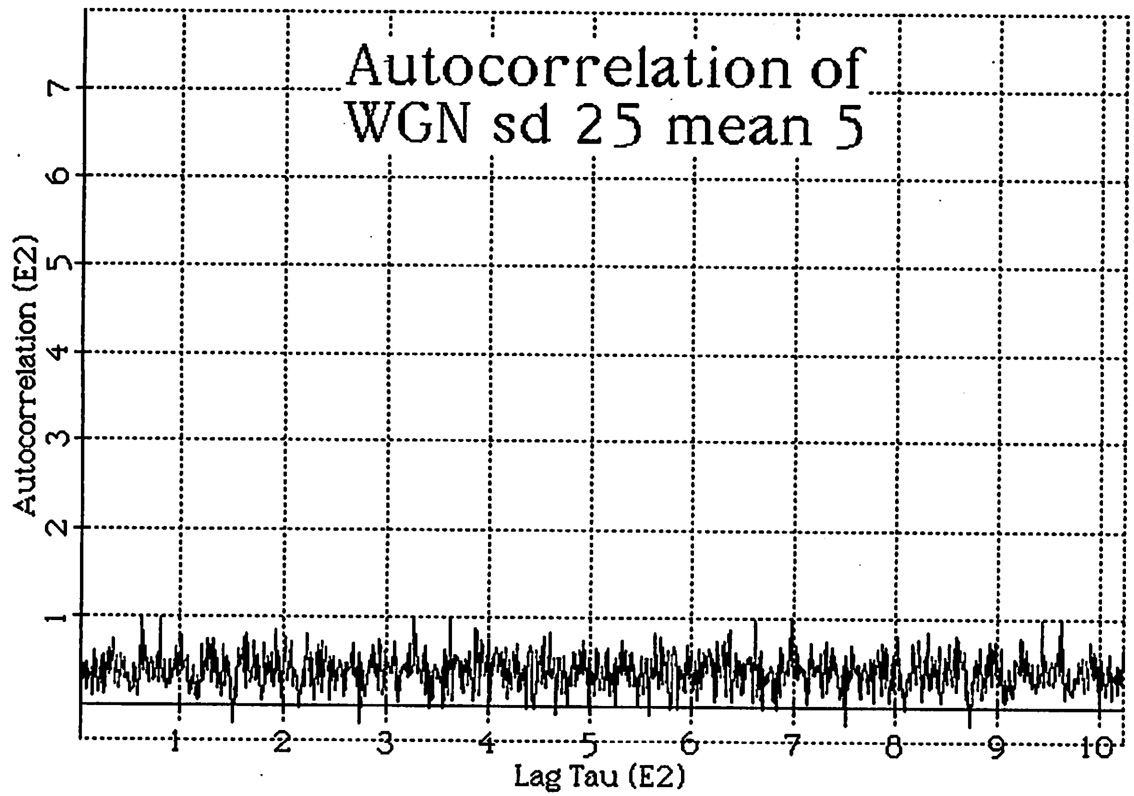
THUS, THE PERIODOGRAM, WHICH REPRESENTS SAMPLES OF THE POWER SPECTRAL DENSITY TREATED AS A CONTINUOUS FUNCTION AND WHERE THE AREA UNDER THE FUNCTON IS THE POWER IS GIVEN BY:

$$P(f_k) = \frac{1}{NU} |X[k]|^2$$

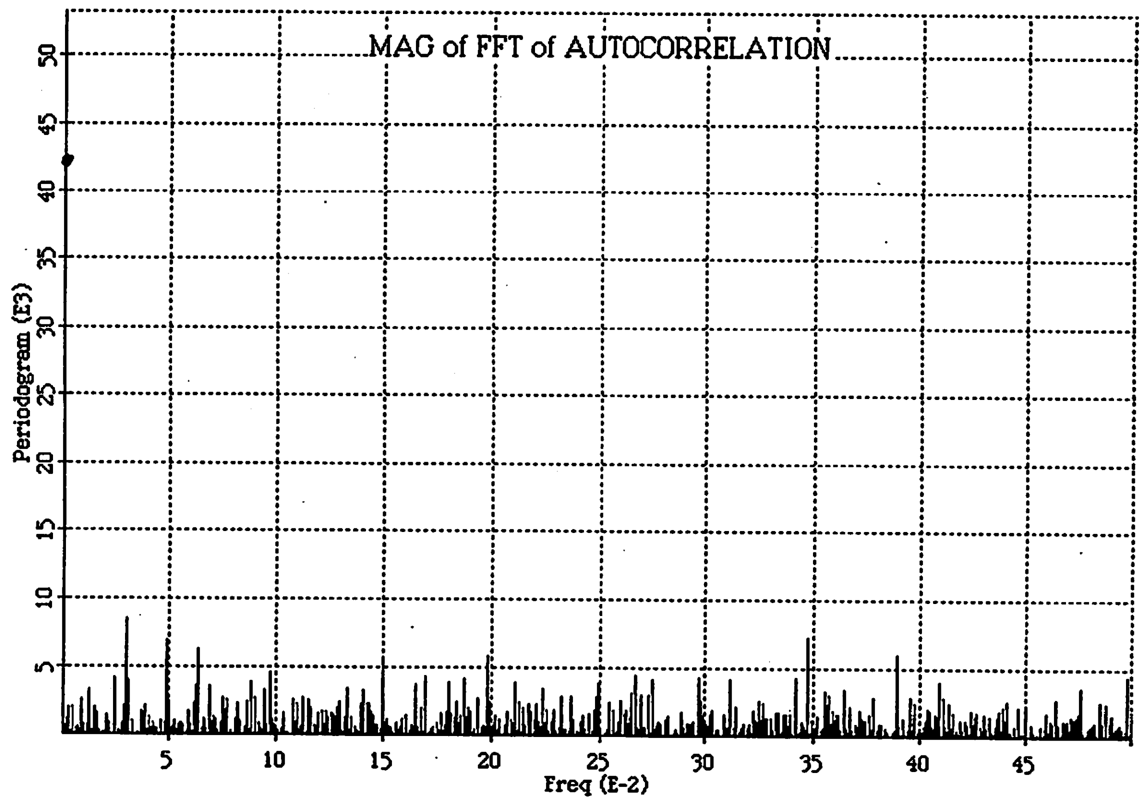
U is shown here as a correction for the type of window used. It is one for the Rectangular Window.

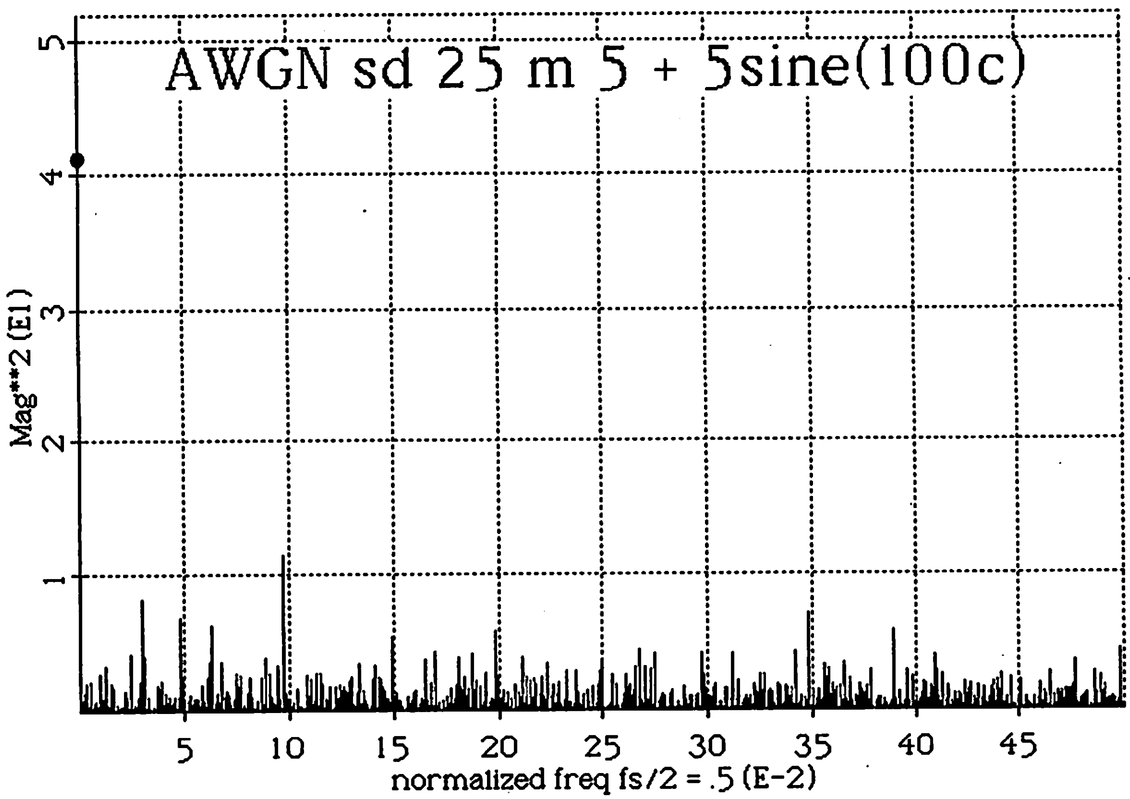

Gauss-Ran-Data(1024pts)
sd-25 m-5
10
5
-5
Rand (E1)
1
2
3
4
5
6
7
8
9
10
Sample (E1)

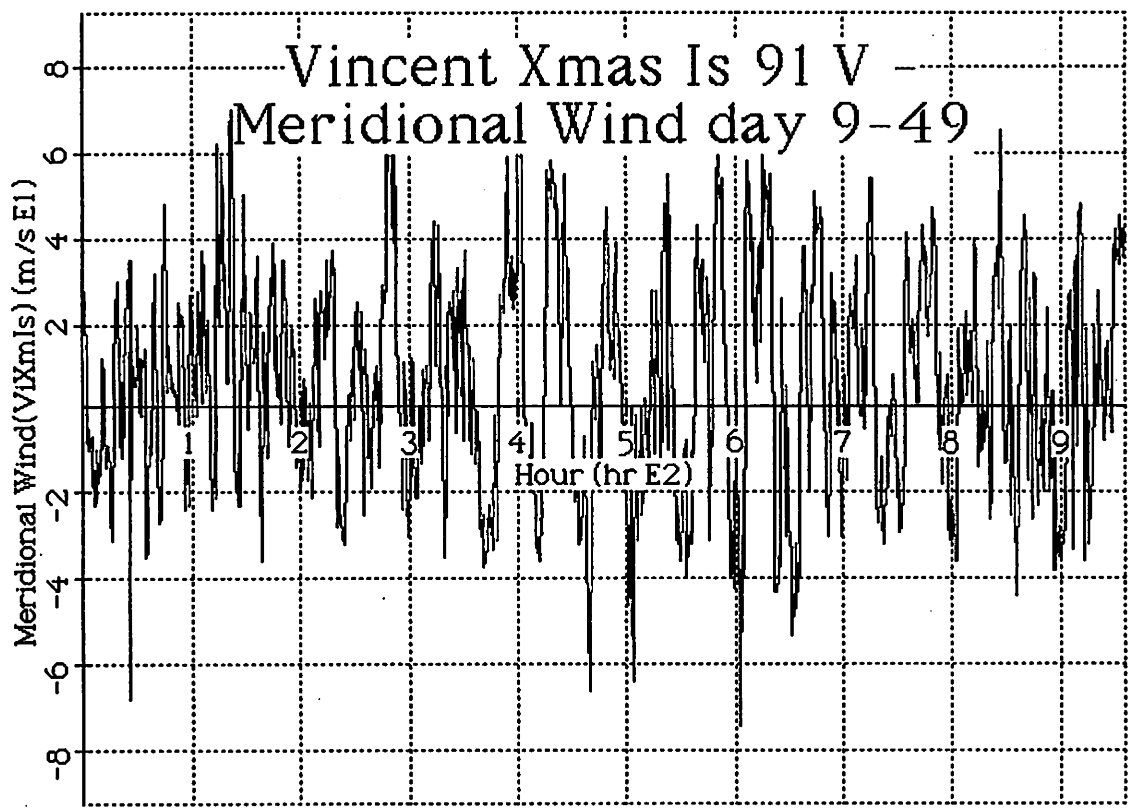

Gaussian Random Data sd 25 m 5
Mag**2 (E1)
1
2
3
4
5
5
10
15
20
25
30
35
40
45
Freq normalized by Sampling rate (c/(time unit) E-2)

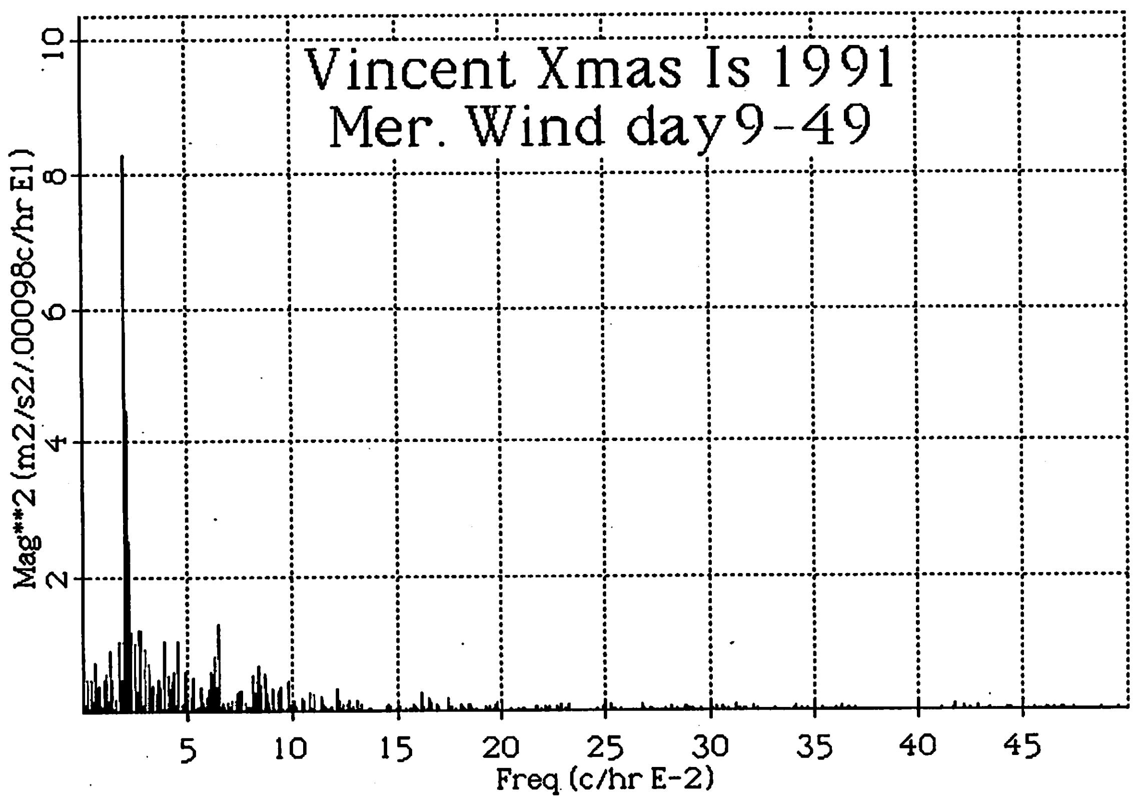


Autocorrelation of
WGN sd 25 mean 5
Autocorrelation (E2)
1
2
3
4
5
6
7
Lag Tau (E2)
1
2
3
4
5
6
7
8
9
10

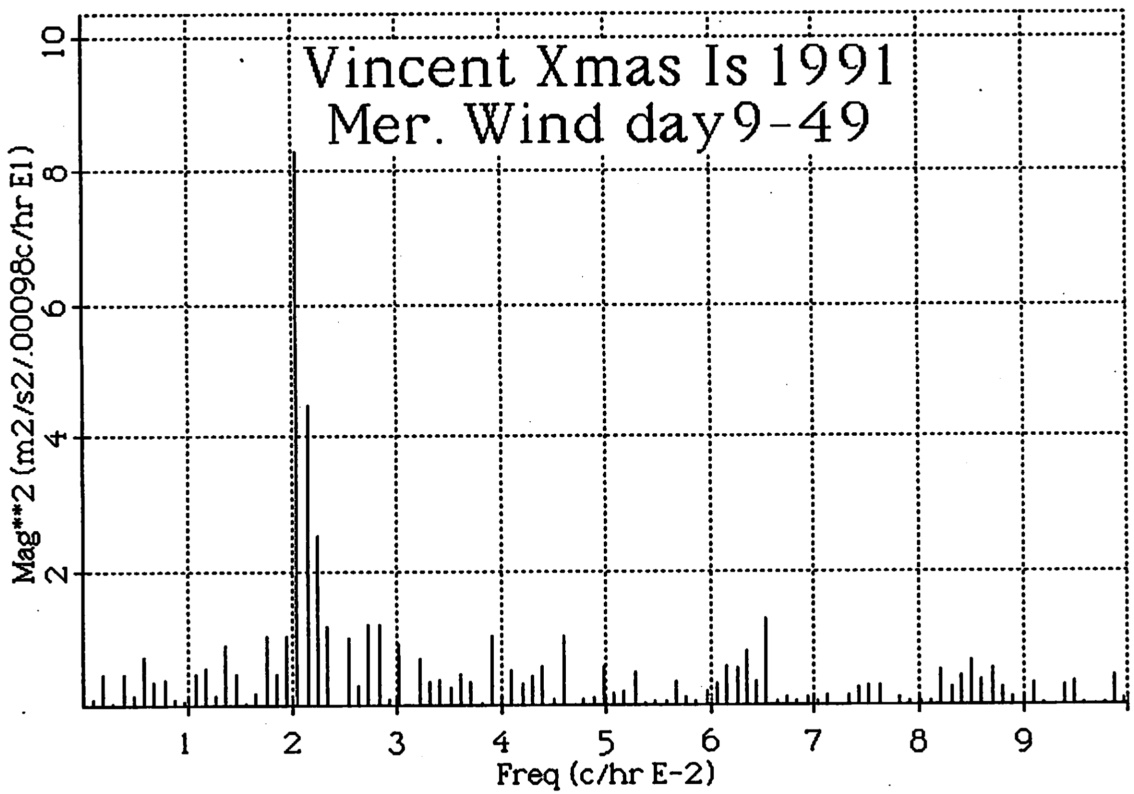

MAG of FFT of AUTOCORRELATION
Periodogram (E3)
50
45
40
35
30
25
20
15
10
5
Freq (E-2)
5
10
15
20
25
30
35
40
45

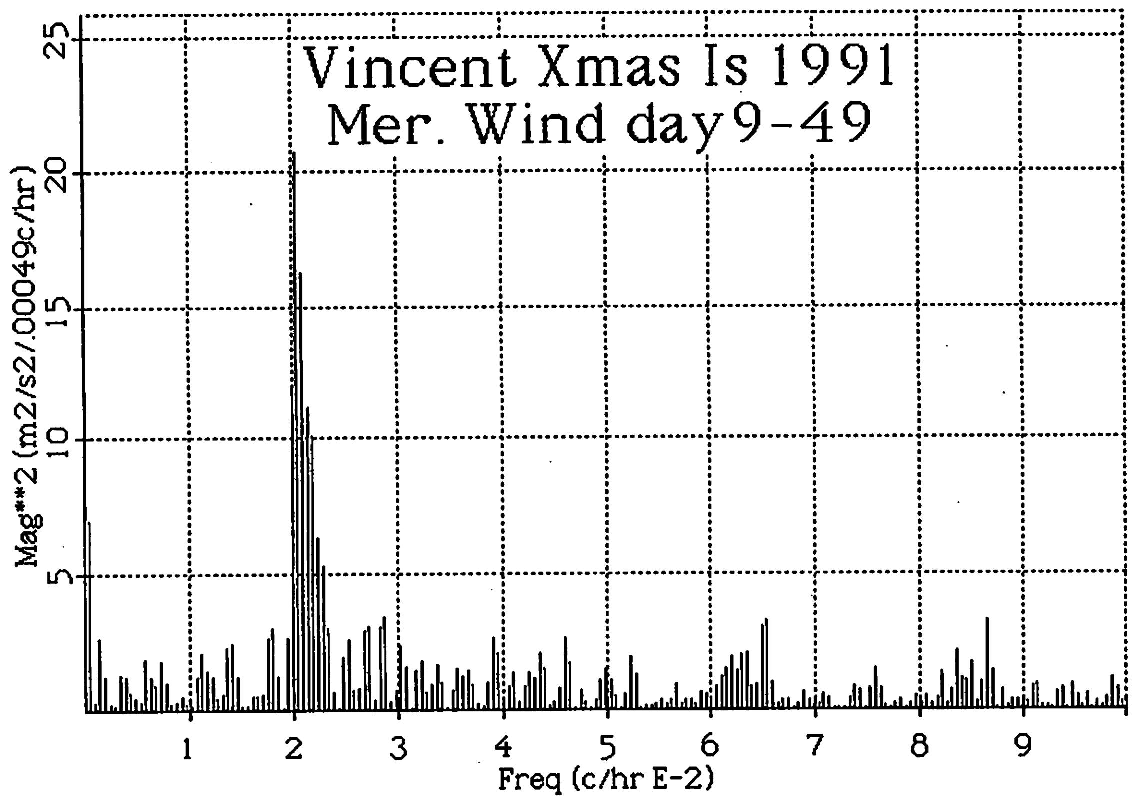

AWGN sd 25 m 5 + 5sine(100c)
Mag**2 (E1)
1
2
3
4
5
5
10
15
20
25
30
35
40
45
normalized freq fs/2 = .5 (E-2)


Vincent Xmas Is 91 V -
Meridional Wind day 9-49
Meridional Wind(ViXmIs) (m/s E1)
Hour (hr E2)
8
6
4
2
-2
-4
-6
-8
1
2
3
4
5
6
7
8
9



960 pts - pad to 1024

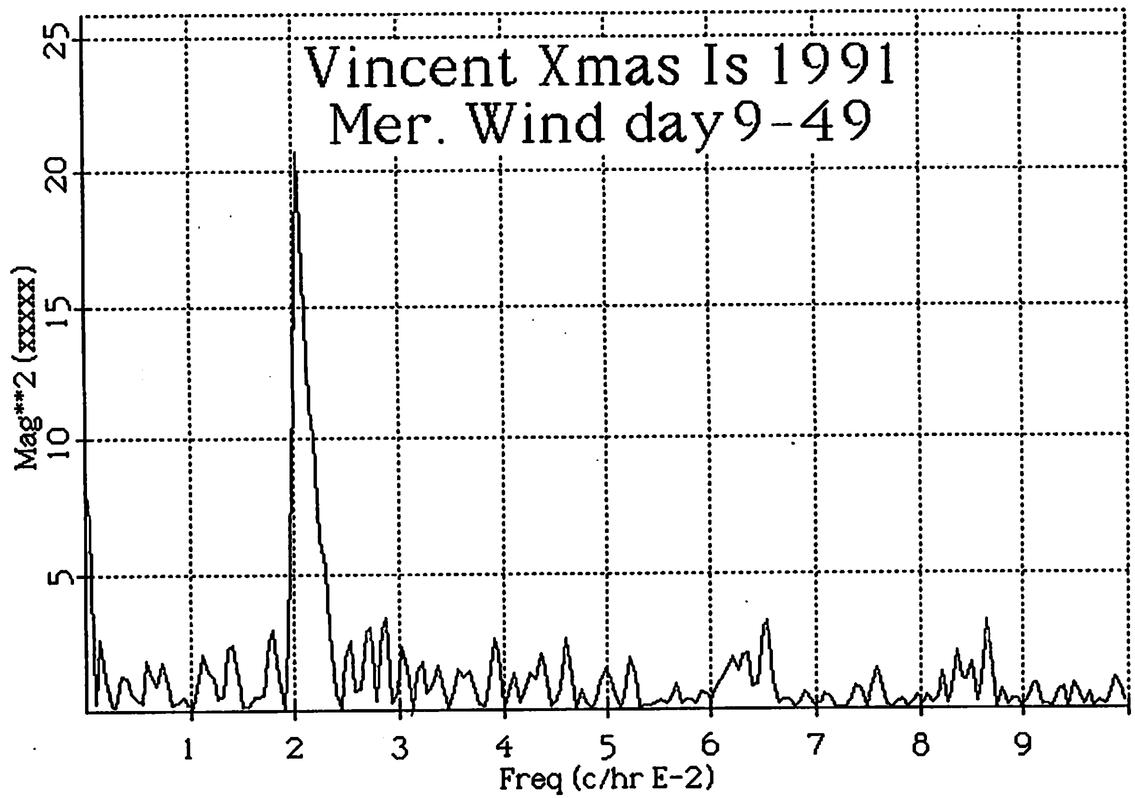

Vincent Xmas Is 1991
Mer. Wind day9-49
Mag**2 (m2/s2/.00098c/hr E1)
10
8
6
4
2
1
2
3
4
5
6
7
8
9
Freq (c/hr E-2)



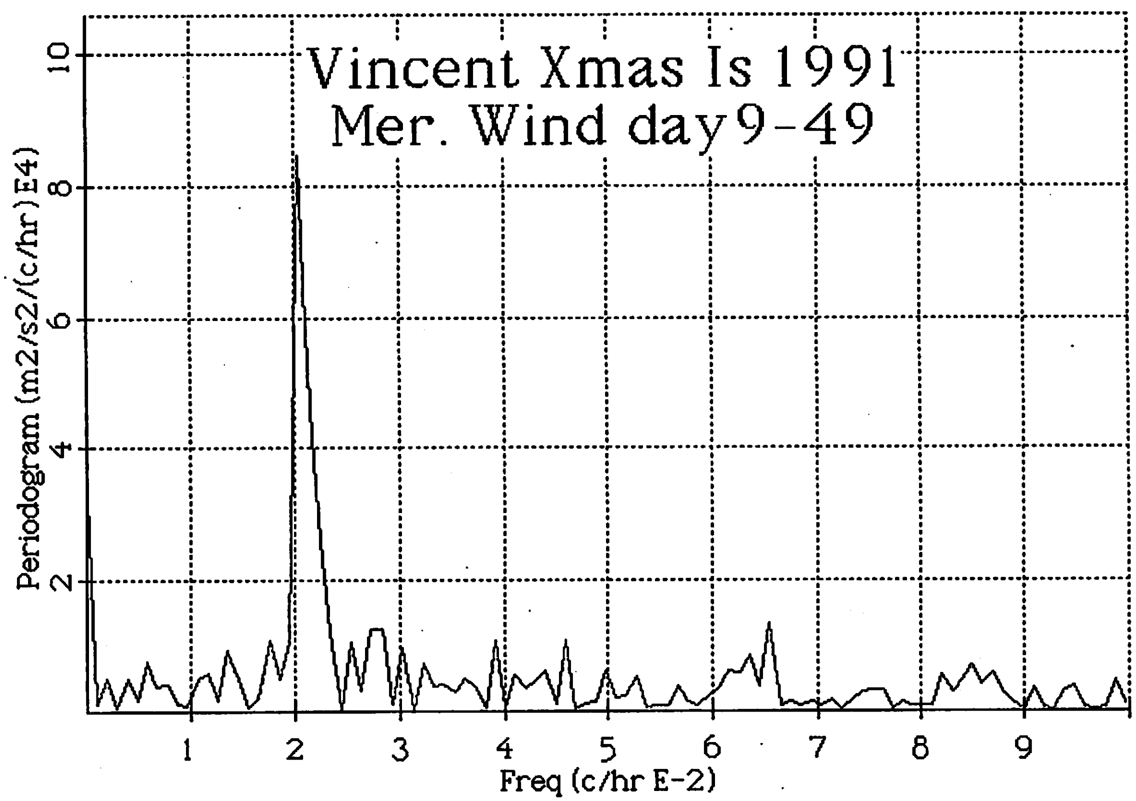
960 pts – pad to 2048

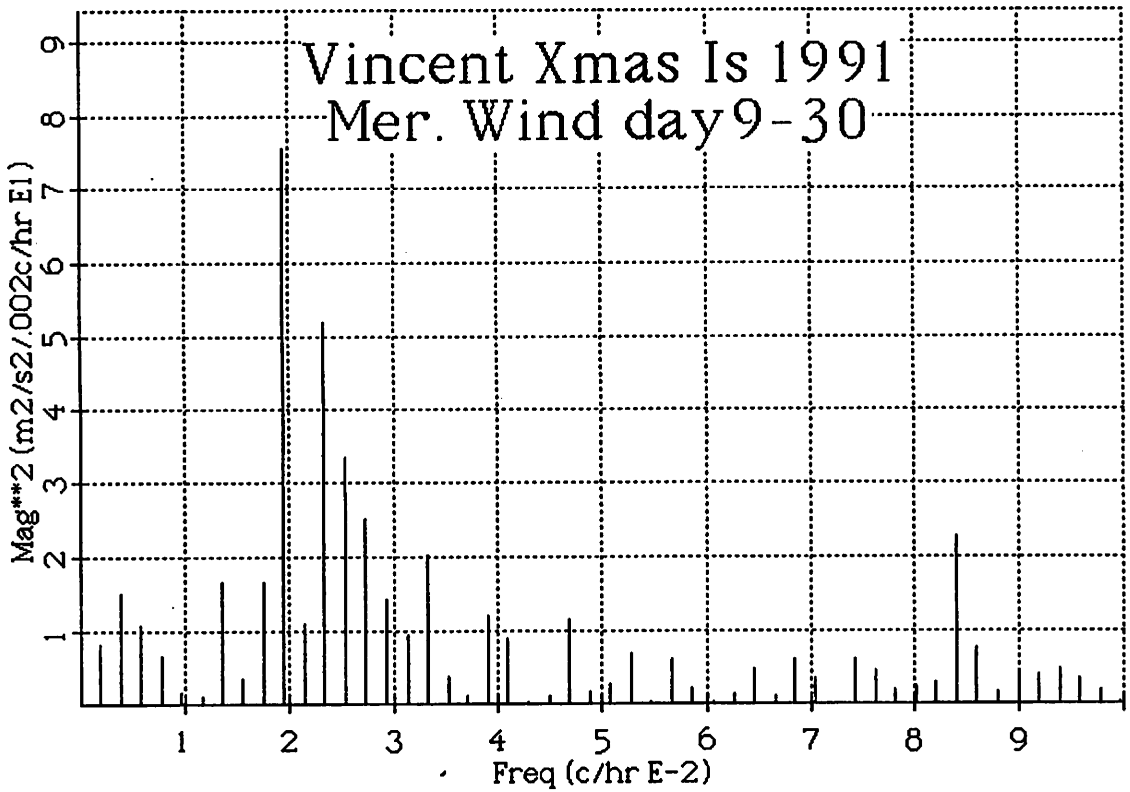
W-1

?



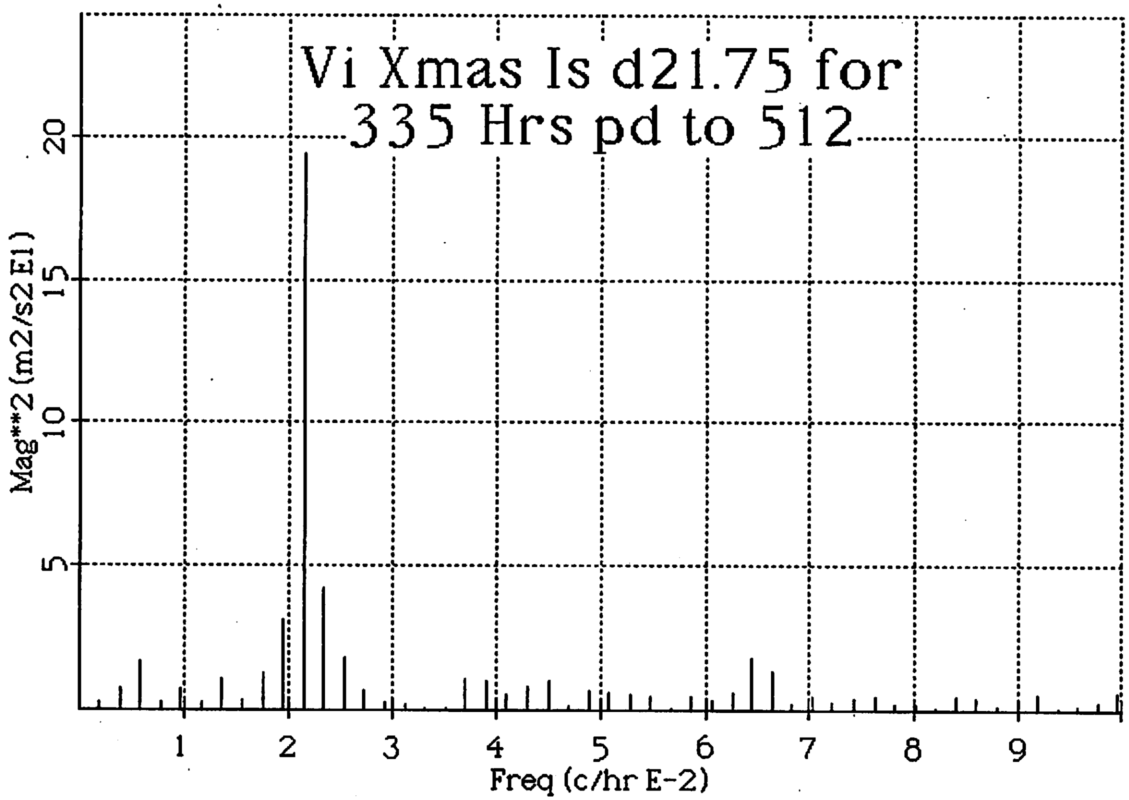
Vincent Xmas Is 1991
Mer. Wind day9-49
Mag**2 (xxxxx)
25
20
15
10
5
1
2
3
4
5
6
7
8
9
Freq (c/hr E-2)


Vincent Xmas Is 1991
Mer. Wind day9-49
Periodogram (m2/s2/(c/hr) E4)
10
8
6
4
2
1
2
3
4
5
6
7
8
9
Freq (c/hr E-2)



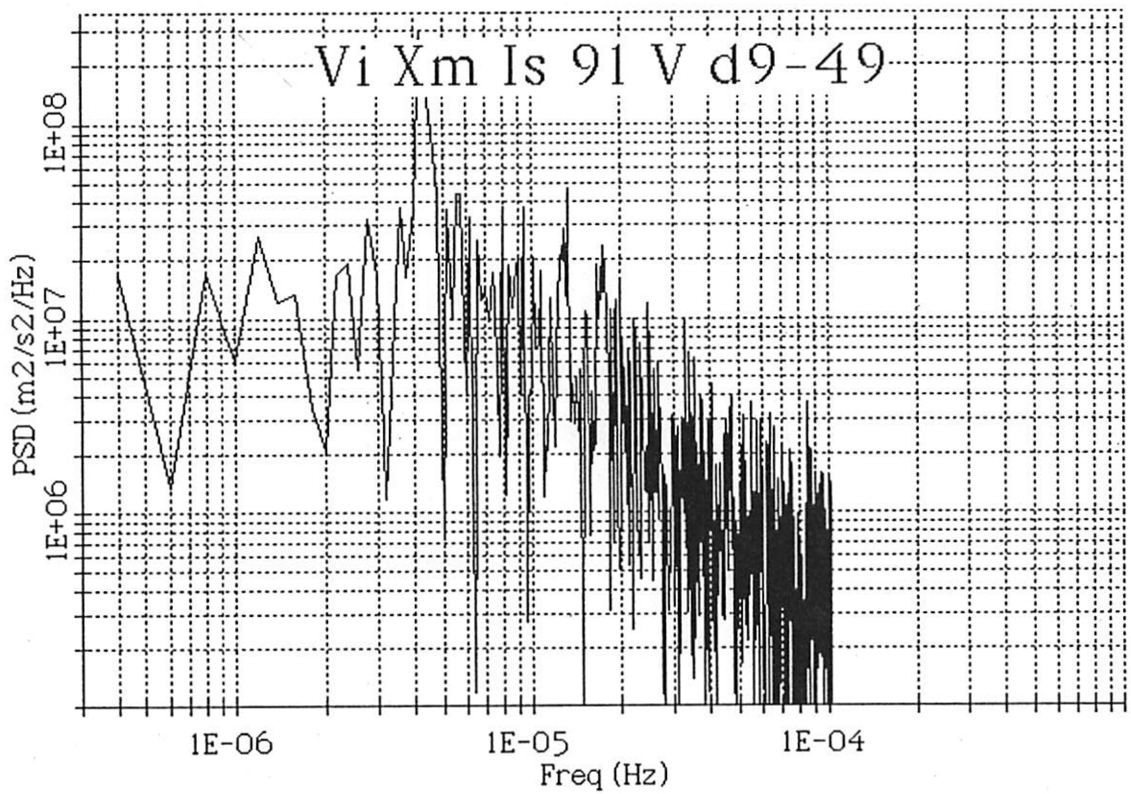
512 hourly pts

W



$$P = \frac{A^2}{2}$$

$$A = \sqrt{190 \times \frac{512}{335} \times 2}$$



960 hourly pts
padded to 1024