UNDERSTANDING DATA ASSIMILATION

APPLICATIONS TO HIGH-LATITUDE IONOSPHERIC ELECTRODYNAMICS

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What is data assimilation?

Combining Information

prior knowledge of the state of system x

empirical or physical models (e.g. physical laws)

complete in space and time

observations y

directly measured or retrieved quantities

incomplete in space and time

Bayes Theorem - Bayesian statistics provides a coherent probabilistic framework for most of DA approaches [e.g., Lorenc, 1986]

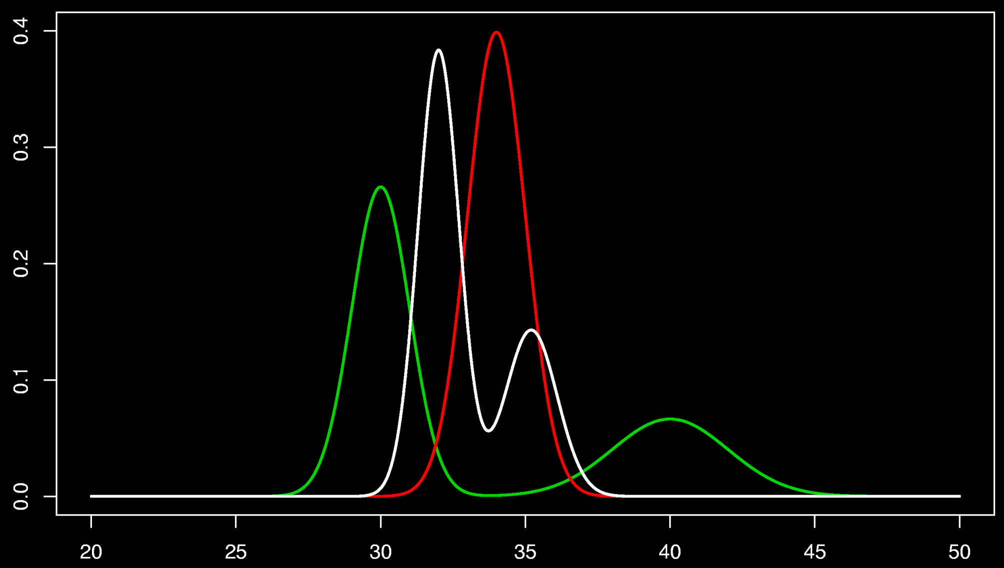
prior $p(\boldsymbol{x})$

observation likelihood $p(\boldsymbol{y}|\boldsymbol{x})$

probability distribution of y when *x have a given value*

posterior

$$p(\boldsymbol{x}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})$$



Bayes Theorem - Bayesian statistics provides a coherent probabilistic framework for most of DA approaches [e.g., Lorenc, 1986]

prior $$p(\boldsymbol{x}) \sim \mathcal{N}(\boldsymbol{x}_b, \mathbf{P}_b) \qquad \boldsymbol{x} = \boldsymbol{x}_b + \boldsymbol{\epsilon}_b$$

observation likelihood $$p(\boldsymbol{y}|\boldsymbol{x}) \sim \mathcal{N}(\mathbf{H}\boldsymbol{x}, \mathbf{R}) \qquad \boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{\epsilon}_y$$

probability distribution of y when *x have a given value*

posterior

$$p(\boldsymbol{x}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})$$

$$p(\boldsymbol{x}|\boldsymbol{y}) \sim \mathcal{N}(\boldsymbol{x}_a, \mathbf{P}_a)$$ where

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{K}(\mathbf{y} - \mathbf{H}\mathbf{x}_b)$$

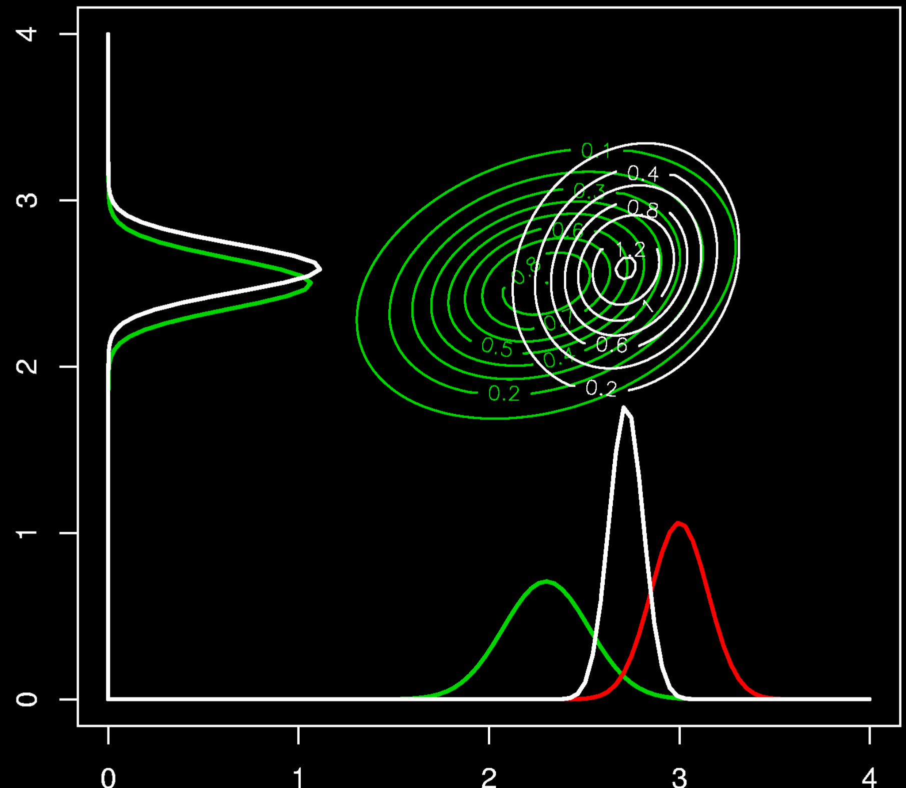
$$\mathbf{P}_a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_b$$

$$\mathbf{K} = \mathbf{P}_b\mathbf{H}^T(\mathbf{H}\mathbf{P}_b\mathbf{H}^T + \mathbf{R})^{-1}$$

What is covariance? – two variables case

Bayes theorem

$$p(\boldsymbol{x}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})$$



prior

$$p(\boldsymbol{x}) \sim \mathcal{N}(\boldsymbol{x}_b, \mathbf{P}_b)$$

$$\boldsymbol{x}_b = (2.3 \quad 2.5)$$

$$\mathbf{P}_b = \begin{pmatrix} 0.225 & 0.05 \\ 0.05 & 0.15 \end{pmatrix}$$

observation-likelihood

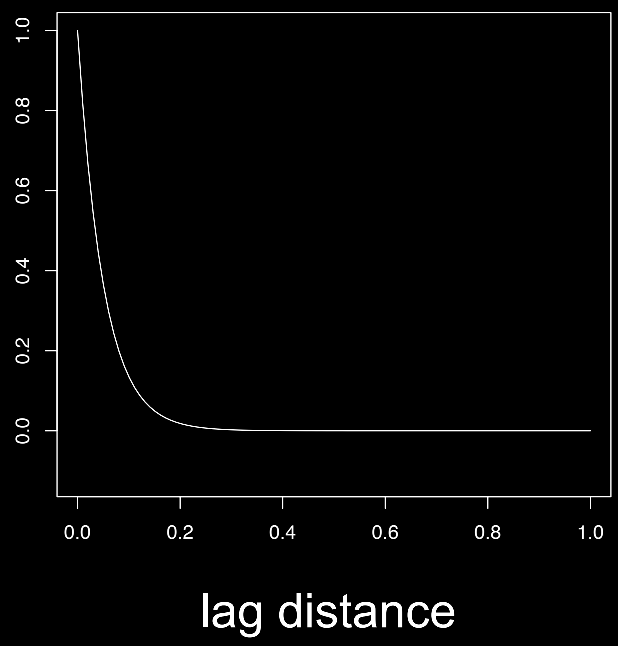
$$p(\boldsymbol{y}|\boldsymbol{x}) \sim \mathcal{N}(\mathbf{H}\boldsymbol{x}, \mathbf{R})$$

$$\mathbf{H} = (1 \quad 0)$$

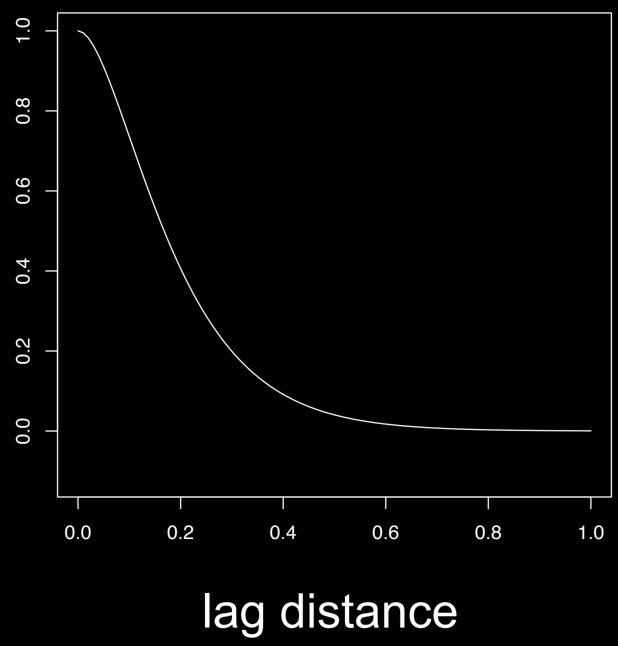
$$x_1 : observed$$

$$x_2 : unobserved$$

What is covariance? – in spatial sense



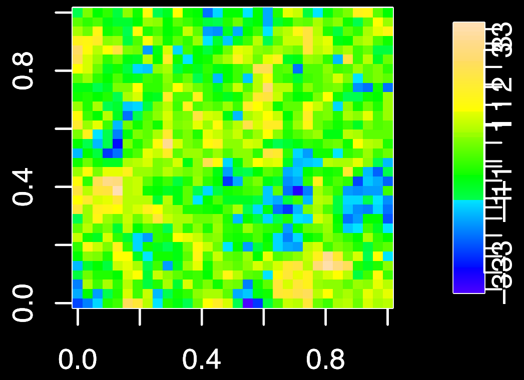
$$\mathbf{P}_1(\mu = 0.5, \gamma = 0.05)$$



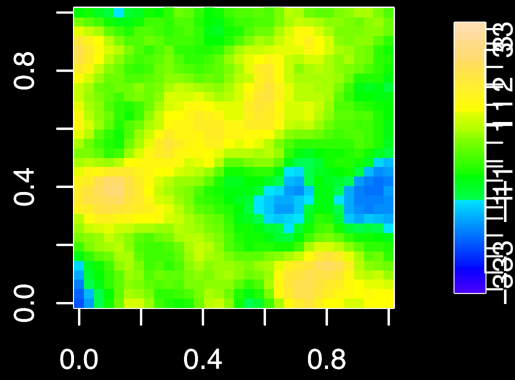
$$\mathbf{P}_2(\mu = 1.5, \gamma = 0.1)$$

What is covariance? – in spatial sense

$\mathbf{x}_1 \sim \mathcal{N}(0, \mathbf{P}_1)$

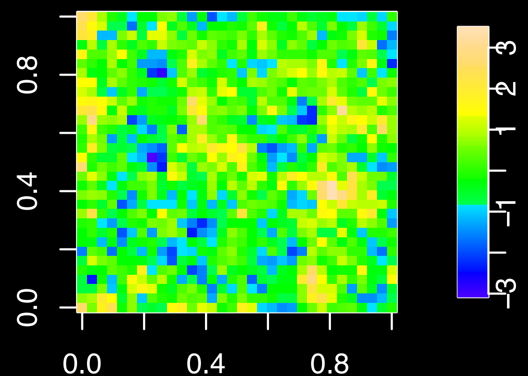
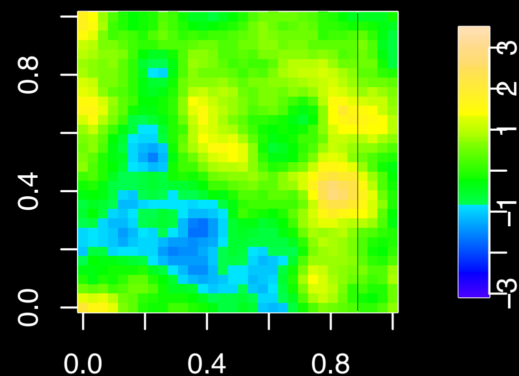


$\mathbf{x}_2 \sim \mathcal{N}(0, \mathbf{P}_2)$

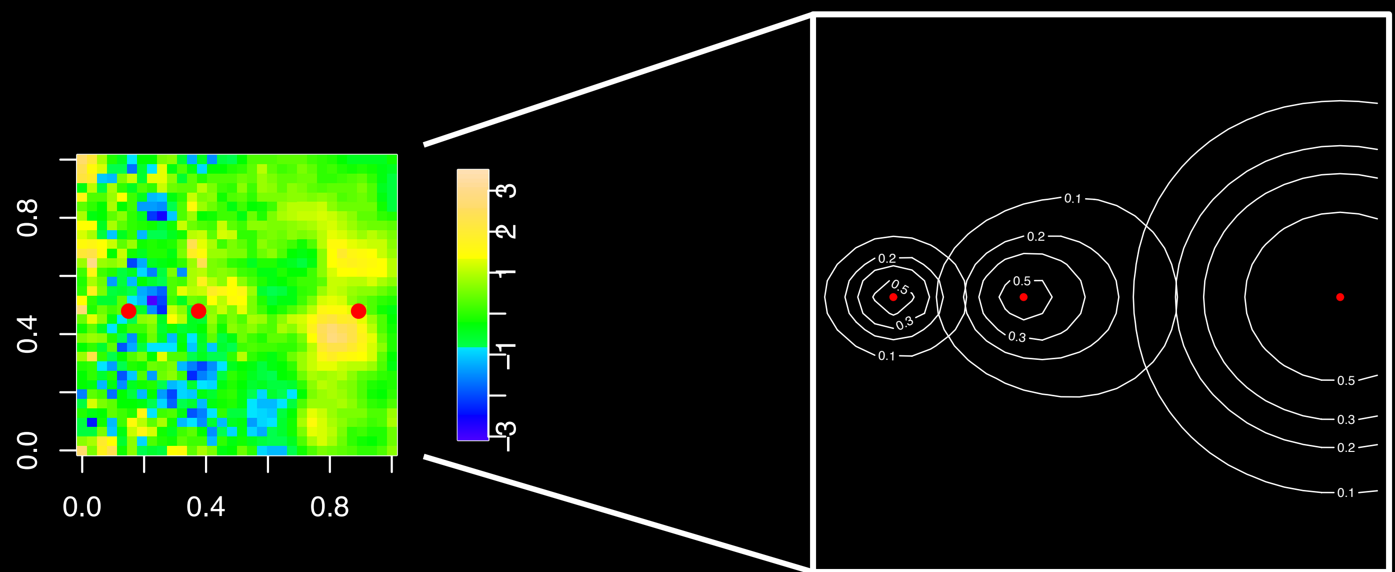


What is covariance? – in spatial sense

$\mathbf{x}_1 \sim \mathcal{N}(0, \mathbf{P}_1)$ $\mathbf{x}_2 \sim \mathcal{N}(0, \mathbf{P}_2)$

2-D Correlation Function



Assimilative Mapping of Ionospheric Electrodynamics

[Richmond and Kamide, 1988]

Inverse procedure to infer maps of

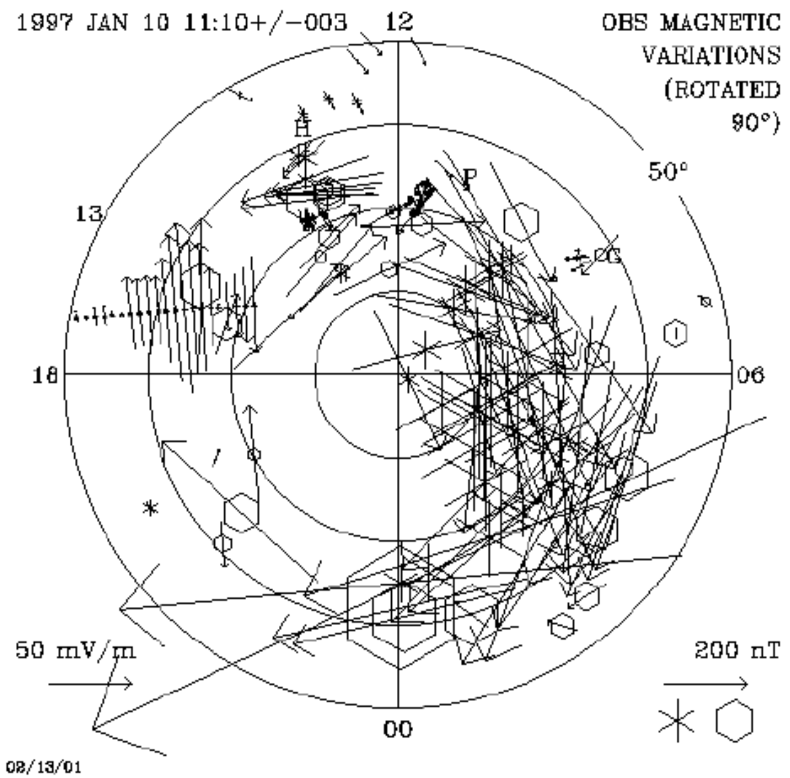
$$\vec{E}, \Phi, \vec{I}_{\perp}, \vec{J}_{//}, \Delta\vec{B}$$

From observations of

$\vec{E}$ IS or HF radar, Satellites

$\vec{I}_{\perp}$ IS radar

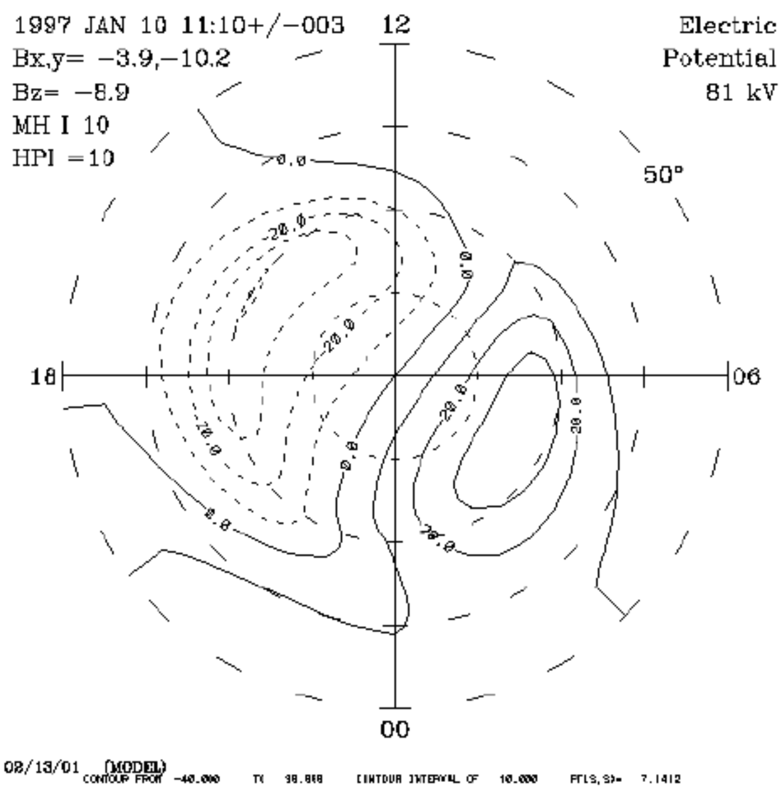
$\vec{J}_{//}$, $\Delta\vec{B}$ Satellite or ground-based magnetometers



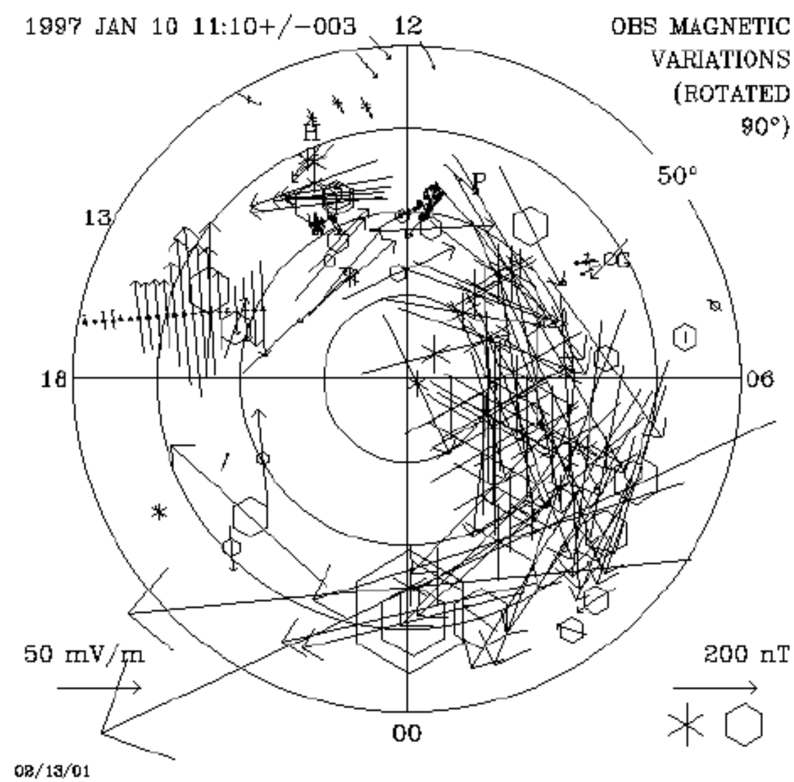
Assimilative Mapping of Ionospheric Electrodynamics

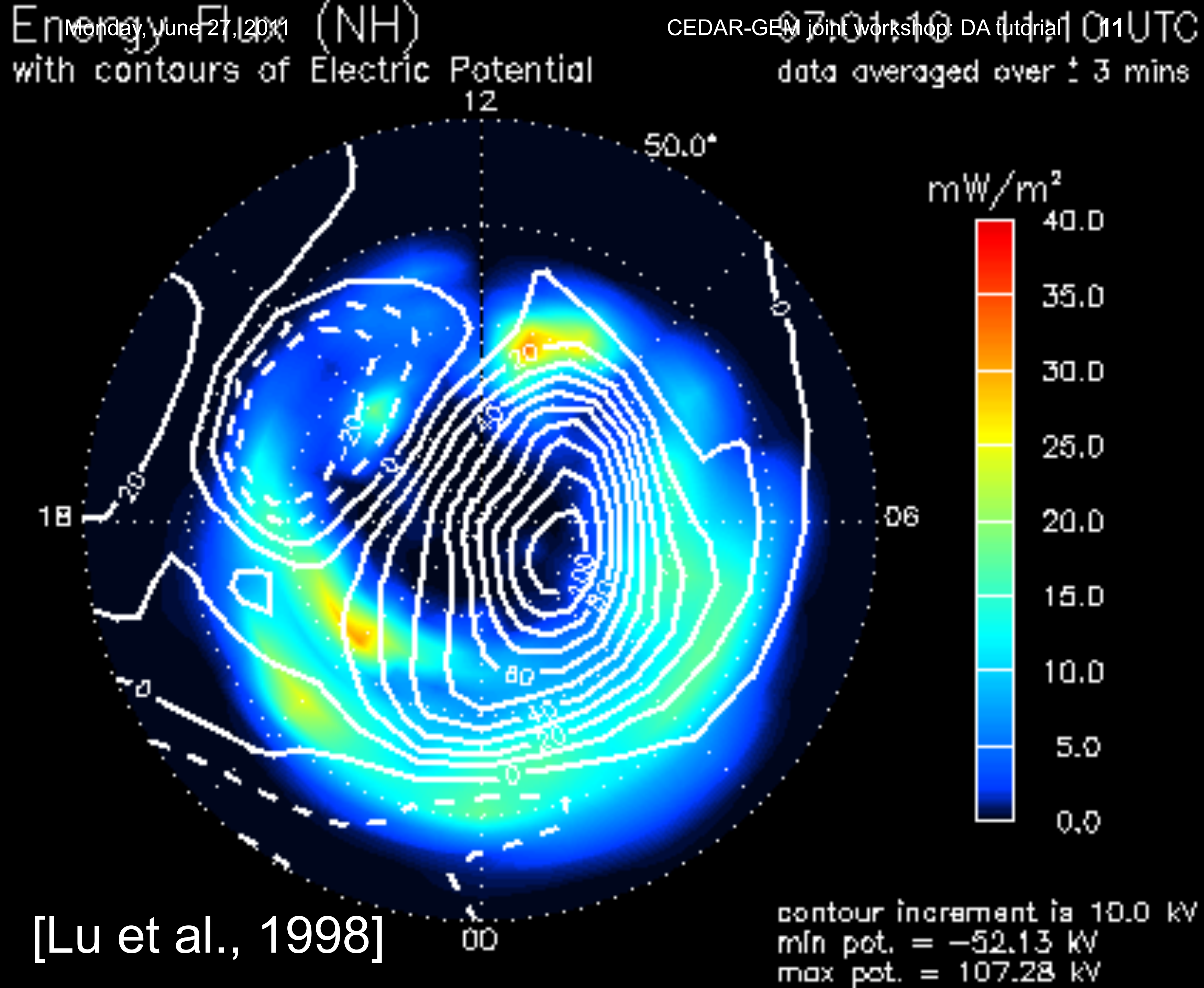
[Richmond and Kamide, 1988]

prior

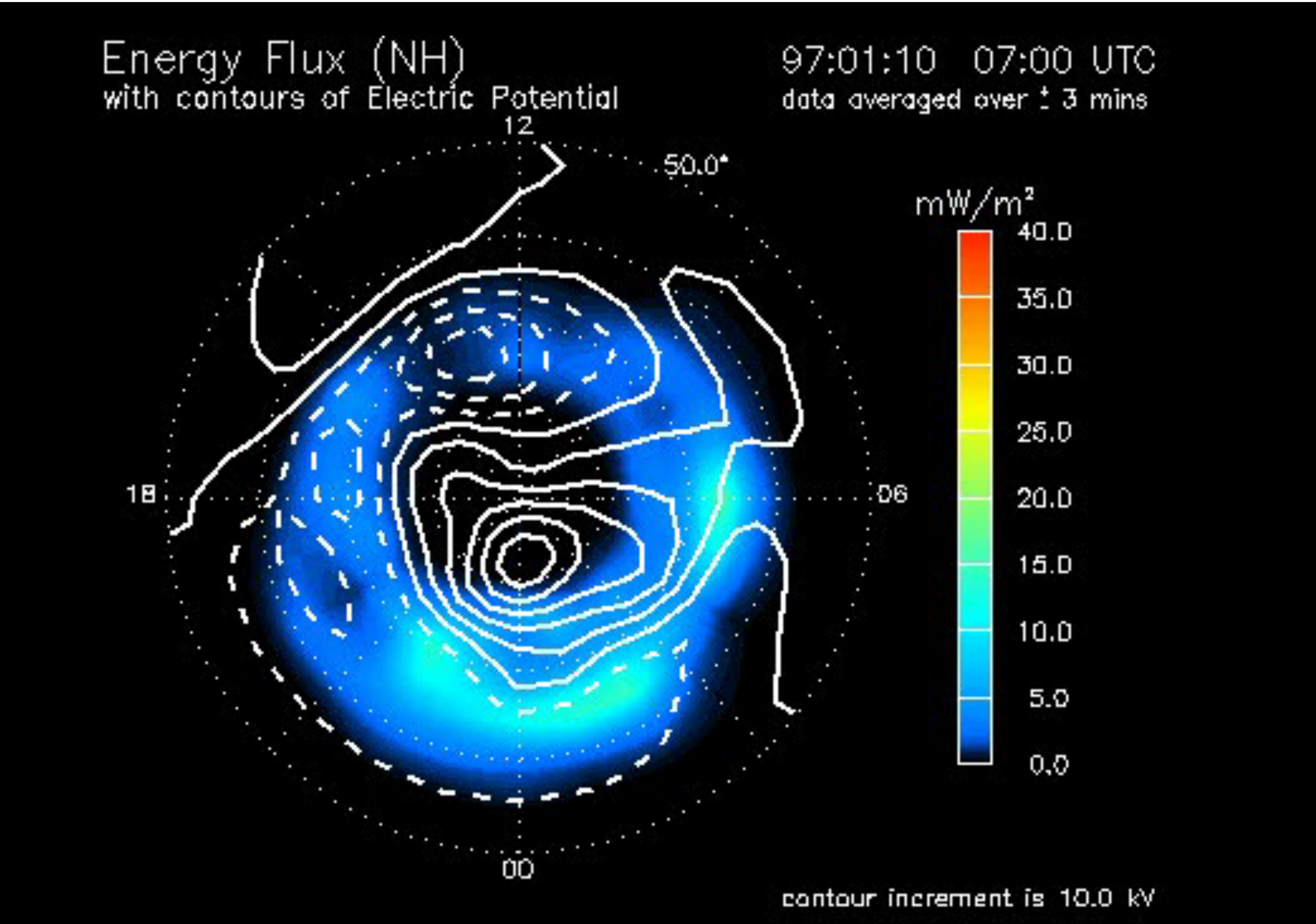


observations




Energy Flux (NH)
with contours of Electric Potential
data averaged over ± 3 mins
12
50.0°
18
06
00
mW/m²
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
contour increment is 10.0 kV
min pot. = −52.13 kV
max pot. = 107.28 kV

[Lu et al., 1998]


Energy Flux (NH)
with contours of Electric Potential
97:01:10 07:00 UTC
data averaged over ± 3 mins
12
50.0°
18
06
00
mW/m²
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
contour increment is 10.0 kV

AMIE – relationship among electromagnetic variables

Inverse procedure to infer maps of

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From observations of

$\vec{E}$ IS or HF radar, Satellites

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linear relationship (for a given $\underline{\underline{\Sigma}}$)

$$F(\vec{E}) = \Phi, \vec{I}_{\perp}, \vec{J}_{//}, \Delta\vec{B}$$

$$\vec{E} = -\nabla\Phi$$

$$\vec{I}_{\perp} = \underline{\underline{\Sigma}} \cdot \vec{E}$$

$$\vec{J}_{||} = \nabla \cdot \vec{I}_{\perp}$$

$$\vec{I}_{\perp}, \vec{J}_{||} \longleftrightarrow \Delta\vec{B}$$

Biot-Savart's law

AMIE – basis functions as forward operator

functional analysis

$$\Phi = \Psi \, x$$

Ψ : spherical harmonics

x : coefficients

$$\vec{E} = -\nabla\Psi \, x$$

forward operator

$$y = \mathbf{H}x$$

$$= F(-\nabla\Psi) \, x$$

linear relationship (for a given $\underline{\underline{\Sigma}}$)

$$F(\vec{E}) = \Phi, \vec{I}_{\perp}, \vec{J}_{//}, \Delta\vec{B}$$

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Biot-Savart's law

Ideas to improve AMIE: Adaptive covariance

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{K}(\mathbf{y} - \mathbf{H}\mathbf{x}_b)$$

$$\mathbf{K} = \mathbf{P}_b(\alpha)\mathbf{H}^T(\mathbf{H}\mathbf{P}_b(\alpha)\mathbf{H}^T + \mathbf{R})^{-1}$$

Maximum likelihood Method *[Dee 1995, Matsuo et al., 2005]*

$$\mathbf{d} = \mathbf{y} - \mathbf{H}\mathbf{x}_b$$

$$\mathbf{d} \sim \mathcal{N}(0, \mathbf{S})$$

$$\text{where} \quad \mathbf{S}(\alpha) = \mathbf{R} + \mathbf{H}\mathbf{P}_b(\alpha)\mathbf{H}^T$$

Find alpha that maximizes the following pdf

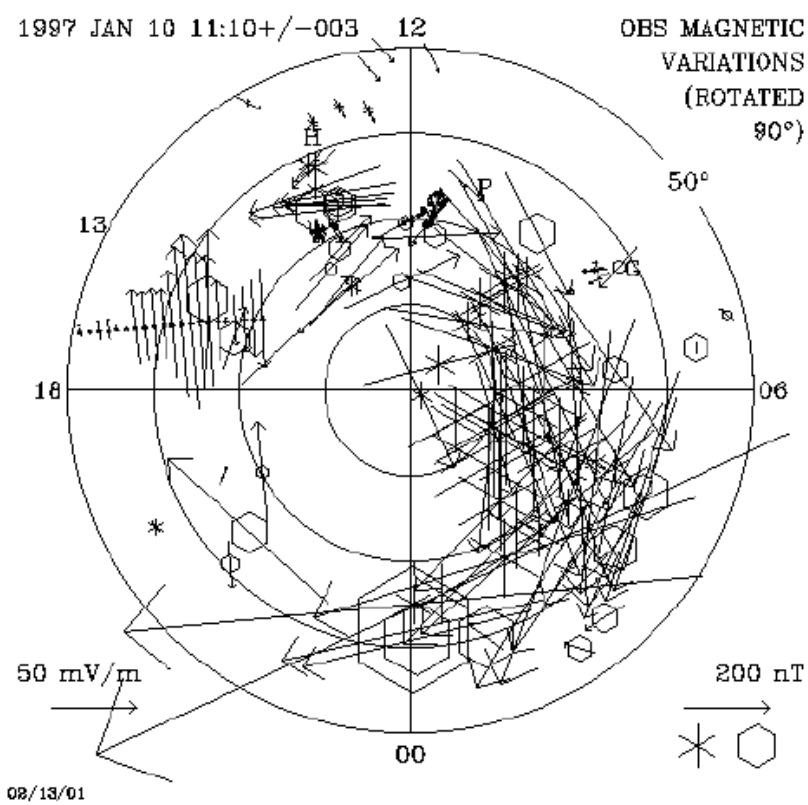
$$p(\mathbf{d}|\alpha) = \frac{1}{\sqrt{2\pi}^{\mathcal{J}}\sqrt{\det \mathbf{S}(\alpha)}} \exp\left[-\frac{\mathbf{d}^T\mathbf{S}^{-1}(\alpha)\mathbf{d}}{2}\right]$$

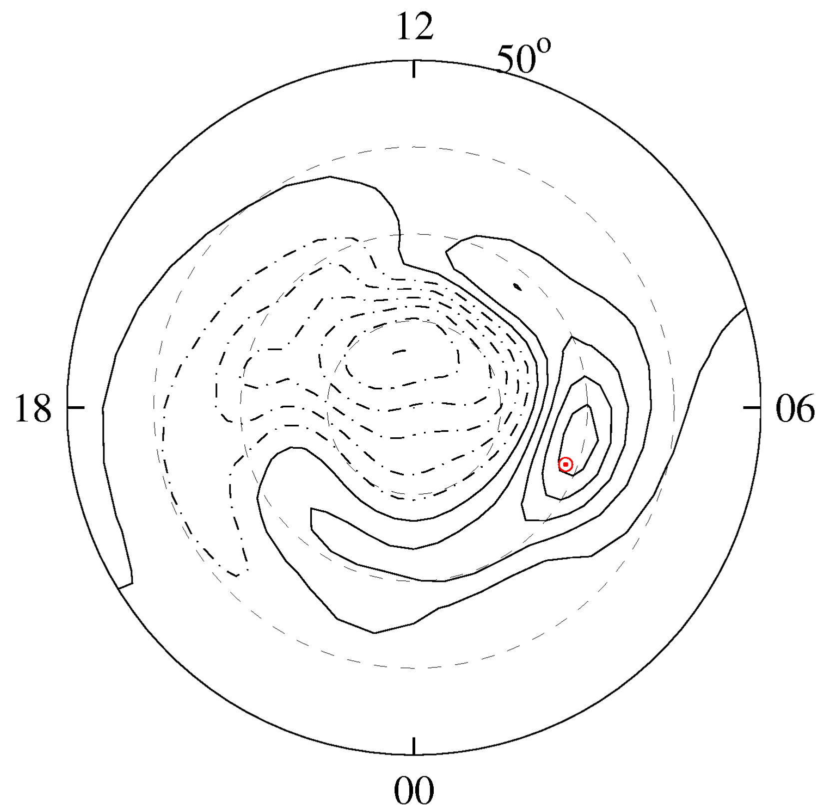
Ideas to improve AMIE: Adaptive covariance

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{K}(\mathbf{y} - \mathbf{H}\mathbf{x}_b)$$

$$\mathbf{K} = \underline{\mathbf{P}_b(\alpha)\mathbf{H}^T}(\mathbf{H}\mathbf{P}_b(\alpha)\mathbf{H}^T + \mathbf{R})^{-1}$$

For given observations

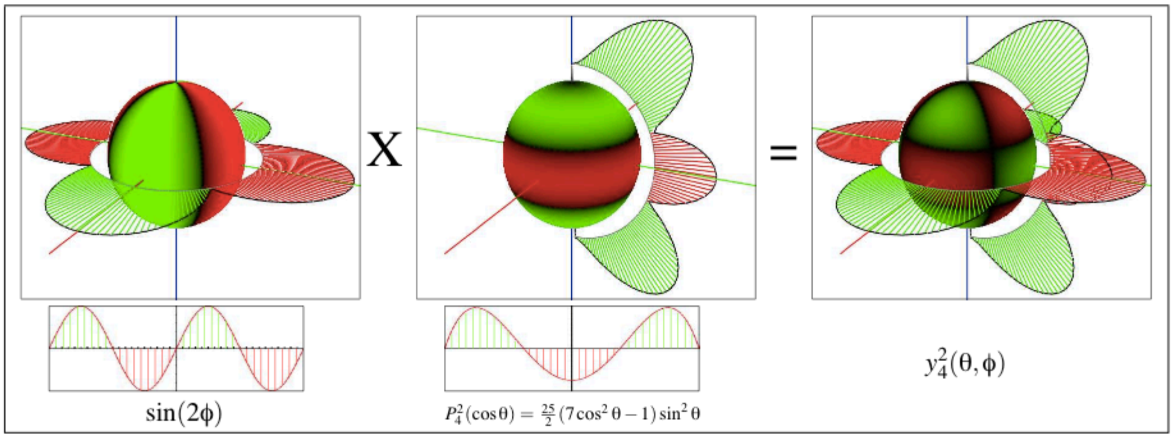




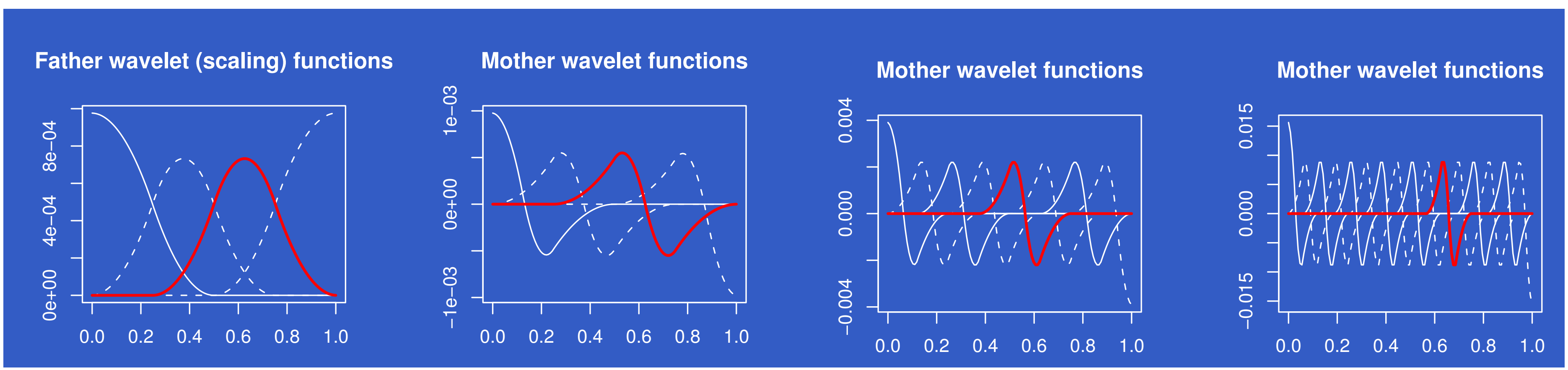
Ideas to improve AMIE: Multi-resolution basis functions

Spherical Harmonics

$$Y_l^m(\theta,\phi) = N_l^{|m|} P_l^{|m|}(\cos\theta)(\cos(m\phi) + i\sin(m\phi))$$

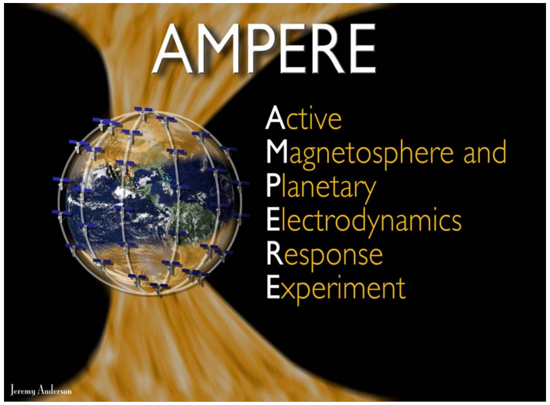


Wavelets



Ideas to improve AMIE: what to minimize $\Delta \mathrm{B}$ or $\mathbf{E}$

How to take advantage of new space-based observations?



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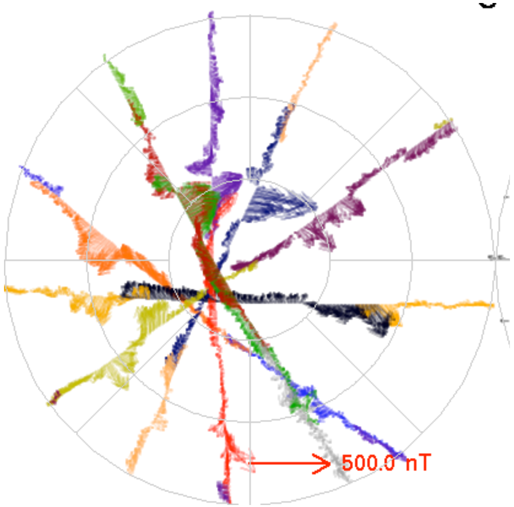
$\Delta\vec{B}$



Summary

- Bayesian statistics as an overarching framework for many of DA methods
- Assumptions: Gaussian distribution, Linear H…
- Role of Covariance in spatial interpolation

- Applications to high-latitude electrodynamics (AMIE)
- Functional analysis of $\mathbf{E}, \mathbf{\Phi}, \mathbf{I}, \mathbf{J}, \Delta\mathbf{B}$
- Kalman update of spherical hamonics coefficients
- Current issues
  - Resolution: global function ->> compactly supported functions
  - New space-based magnetometer observations ->> $\mathbf{E}$ or $\Delta\mathbf{B}$
  - Improve conductance models
  - Adaptive Covariance